Modeling, Analysis, & Control of Compliantly-Coupled Systems

Dr. Kevin Craig
Professor of Mechanical Engineering
Rensselaer Polytechnic Institute
Topics

• **Introduction:**
  - Parasitic Effects
  - Resonance / Anti-Resonance in Dynamic Systems

• **Classification of Resonance**
  - Low-Frequency Resonance
  - High-Frequency Resonance

• **Curing Resonance**
  - Mechanical Cures
  - Filters: Low-Pass, Notch, Bi-Quad
  - Observer-Based Techniques: Acceleration Feedback, Observer Filtering, Active Resonance Damping
Parasitic Effects

- Parasitic effects are present in all real-world systems and are troublesome to account for when the systems are designed. They are rarely disabling, but are debilitating if not dealt with effectively.

- These effects include:
  - Coulomb Friction
  - Time Delay
  - Unmodeled Resonances
  - Saturation
  - Backlash
  - Nonlinearity
  - Noise

Our focus here!
• **Questions:**
  – Are they significant?
  – What to do about them?

• **Approaches:**
  – Ignore them and hope for the best! Murphy’s Law says ignore them at your own peril.
  – Include the parasitic effects that you think may be troublesome in the truth model of the plant and run simulations to determine if they are negligible.
  – If they are not negligible and can adversely affect your system, you need to do something – but what?
• **General Remedies include:**
  – Include a model of the parasitic effect in the observer of the plant in observer-based controllers.
  – Alter the design to reduce the effective loop gain of the controller, especially at high frequencies where the effects of parasitics are often predominant. This generally entails sacrifice in performance.
  – Techniques specifically intended to enhance robustness of the design are also likely to be effective, but may entail use of a more complicated control algorithm.
• **Linear Time-Invariant (LTI) Systems**
  – A frequency-domain transfer function is limited to describing elements that are linear and time invariant – severe restrictions! No real-world system meets them!
  – Linear Time-Invariant System Properties
    • Homogeneity
    • Superposition
    • Time Invariance
  – Transfer Functions, the basis of classical control theory, require LTI systems, but no real-world system is completely LTI. There are regions of nonlinear operation and often significant parameter variation.
– Examples of Linear Behavior
  • addition, subtraction, scaling by a fixed constant, integration, differentiation, time delay, and sampling
– No practical control system is completely linear and most vary over time.
– LTI systems can be represented completely in the frequency domain. Non-LTI systems can have Bode plots, however, these plots change depending on system operating conditions.
• **Non-LTI Behavior**
  – Non-LTI behavior is any behavior that violates one or more of the three criteria for an LTI system.
  – Within nonlinear behavior, an important distinction is whether the variation is **slow or fast** with respect to the loop dynamics.

• **Slow Variation**
  – When the variation is slow, the nonlinear behavior may be viewed as a linear system with parameters that vary during operation.
  – The dynamics can still be characterized effectively with a transfer function. However, the Bode plots will change at different operating points.
• **Fast Response**
  – If the variation of the loop parameter is fast with respect to the loop dynamics, the situation becomes more complicated. Transfer functions cannot be relied upon for analysis.
  – The definition of fast depends on the system dynamics.

• **Fast vs. Slow**
  – The line between fast and slow is determined by comparing the rate at which the parameter changes to the bandwidth of the control system. If the parameter variation occurs over a period of time no faster than 10 times the control loop settling time, the effect can be considered slow for most applications.

\[ t_{s-2\%} \approx \frac{4}{\omega_{BW}} \]
• **Dealing with Nonlinear Behavior**
  
  – Nonlinear behavior can usually be ignored if the changes in parameter values effect the loop gain by no more than about 25%. A variation this small will be tolerated by systems with reasonable margins of stability.
  
  – If a parameter varies more than that, there are at least three courses of action:
    • Modify the plant
    • Tune for the worst-case conditions
    • Compensate for the nonlinearity of the control loop
• **Modify the Plant**
  
  – Modifying the plant to reduce the parameter variation is the most straightforward solution to nonlinear behavior. It cures the problem without adding complexity to the controller or compromising system performance.

  – This solution is commonly employed in the sense that components used in control systems are generally better (closer to LTI) than components used in open-loop systems.

  – Enhancing the LTI behavior of a loop component can increase its cost significantly. Components for control systems are often more expensive than open-loop components.
• **Tune for the Worst-Case Conditions**
  
  – Assuming that the variation from the non-LTI behavior is slow with respect to the control loop, its effect is to change gains in the control loop. In this case, the operating conditions can be varied to produce the worst-case gains while tuning the control system. Doing so will ensure stability for all operating conditions.

  – Tuning the system for worst-case operating conditions generally implies tuning the proportional gain of the inner loop when the plant gain is at its maximum. This ensures that the inner loop will be stable in all conditions; parameter variation will only lower the loop gain, which will reduce responsiveness but will not cause instability.
– The other loop gains (inner loop integral and the outer loops) should be stabilized when the plant gain is minimized. This is because minimizing the plant gain reduces the inner loop response; this will provide the maximum phase lag to the outer loops and again provides the worst case for stability.

– So tune the proportional gain with a high plant gain and tune the other gains with a low plant gain to ensure stability in all conditions.

– The penalty for tuning for worst case is the reduction in responsiveness. Consider the proportional gain. Because the proportional term is tuned with the highest plant gain, the loop gain will be reduced at operating points where the plant gain is low.
– In general, you should expect to lose responsiveness in proportion to plant variation.

• **Compensate in the Control Loop**
  – Compensating for the nonlinear behavior in the controller requires that a gain equal to the inverse of the non-LTI behavior be placed in the loop.
  – This is called gain scheduling. By using gain scheduling, the impact of the non-LTI behavior is eliminated from the control loop.
  – Gain scheduling requires that the non-LTI behavior be slow with respect to any transfer functions between the non-LTI component and the scheduled gain.
- This is a less onerous requirement than being slow with respect to the loop bandwidth because the loop components are always much faster than the loop itself.
- Gain scheduling assumes that the non-LTI behavior can be predicted to reasonable accuracy (generally ± 25%) based on information available to the controller. This is often the case.
- Many times, a dedicated control loop will be placed under the direction of a larger system controller. The more flexible system controller can be used to accumulate information on a changing gain and then modify gains inside dedicated controllers to affect the compensation.
The chief shortcoming of gain scheduling via the system controller is limited speed. The system controller may be unable to keep up with the fastest plant variations. Still this solution is commonly employed because of the controller’s higher level of flexibility and broader access to information.
• **Let’s now focus on the case of unmodeled resonances.**

• Accurate modeling of the dynamic behavior of a mechanical system will result in a dynamic system of higher order than you probably would want to use for the design model.

• **For example**, consider a shaft that connects a drive motor to a load. Possibilities include:
  – Shaft has infinite stiffness (rigid)
  – Shaft has a stiffness represented by a spring constant that leads to a resonance in the model
  – Shaft is represented by a Partial Differential Equation that leads to an infinite number of resonances
Physical Modeling

Less Real, Less Complex, More Easily Solved

Truth Model | Design Model

More Real, More Complex, Less Easily Solved

Hierarchy Of Models
Always Ask: Why Am I Modeling?
• In most situations, the frequencies of these resonances will be orders of magnitude above the operating bandwidth of the control system and there will be enough natural damping present in the system to prevent any trouble.
• In applications that require the system to have a bandwidth that approaches the lowest resonance frequency, difficulties can arise.
• A control system based on a design model that does not account for the resonance may not provide enough loop attenuation to prevent oscillation and possible instability at or near the frequency of the resonance.
If the precise nature of the resonances are known, they can be modeled and included in the design model.

However, in many applications the frequencies of the poles (and neighboring zeros) of the resonances are not known with precision or may shift during the operation of the system. A small error in a resonance frequency, damping, or distance between the pole and zero might result in a compensator design that is even worse than a compensator that ignores the resonance phenomenon.
Resonance / Anti-Resonance Behavior in Dynamic Systems

• Mechanical resonance is a pervasive problem in servo systems usually caused by compliance of power-transmission components.

• This compliance often reduces stability margins, forcing gains down and reducing machine performance.

• Servo performance is enhanced when control-law gains are high; however, instability results when a high-gain control law is applied to a compliantly-coupled motor and load.
• Mechanical resonance needs only two inertias coupled by compliant components to manifest itself.

• Machine designers specify transmission components (e.g., couplings, gearboxes) to be rigid in an effort to minimize mechanical compliance. Some compliance is unavoidable. Also, cost and weight limitations force designers to choose lighter-weight components than would otherwise be desirable, leading to low transmission rigidity.

• **Consider Two Cases**: Compliantly-Coupled Motor and Load; Belt-Driven Load with Motor
Two-Mass, Three-Spring Dynamic Physical System: A Mechatronic Teaching System

Review & Summary
Engineering System Investigation Process

Start Here

Physical System

System Measurement

Measurement Analysis

Comparison: Predicted vs. Measured

Is the Comparison Adequate?

Design Changes

NO

YES

Parameter Identification

Physical Model

Mathematical Model

Mathematical Analysis

Engineering System Investigation Process
Physical System Schematic

Two-Mass Three-Spring Dynamic System

Compliantly-Coupled Systems
Diagram of Physical Model

Two-Mass Three-Spring Dynamic System
Physical Model
Nonlinear Equations of Motion

\[ M_{1\text{-eff}} \ddot{x}_1 + B_{1\text{-eff}} \dot{x}_1 + 2Kx_1 = Kx_2 - F_{f1\text{-eff}} \left[ \text{sgn}(\dot{x}_1) \right] + \frac{K_t K_a}{r_p} V_{in} \]

\[ M_{2\text{-eff}} \ddot{x}_2 + B_{2\text{-eff}} \dot{x}_2 + 2Kx_2 = Kx_1 - F_{f2\text{-eff}} \left[ \text{sgn}(\dot{x}_2) \right] \]

\[ M_{1\text{-eff}} = M_1 + \frac{J_T}{r_p^2} \]

\[ B_{1\text{-eff}} = B_1 + \frac{B_m}{r_p^2} \]

\[ F_{f1\text{-eff}} = F_{f1} + \frac{T_f}{r_p} \]
Linear Model: Transfer Functions

\[
\frac{X_1(s)}{V_{\text{in}}(s)} = \frac{\left(\frac{K_t K_a}{r_p}\right)(M_2 s^2 + B_2 s + 2K)}{D(s)}
\]

\[
\frac{X_2(s)}{V_{\text{in}}(s)} = \frac{\left(\frac{K_t K_a}{r_p}\right)K}{D(s)}
\]

\[
D(s) = (M_{1-\text{eff}} M_2) s^4 + (M_{1-\text{eff}} B_2 + M_2 B_{1-\text{eff}}) s^3 + \left[(M_{1-\text{eff}} + M_2) 2K + B_{1-\text{eff}} B_2\right] s^2 + \left[(B_{1-\text{eff}} + B_2) 2K\right] s + 3K^2
\]
Linear Model: State-Space Equations

\[
\begin{align*}
q &= Aq + Bu \\
y &= Cq + Du
\end{align*}
\]

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-2K & -B_{1\text{-eff}} & K & 0 \\
0 & 0 & 0 & 1 \\
K & 0 & -2K & -B_2
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
+ \begin{bmatrix}
0 \\
K_t K_a \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
V_{in}
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{bmatrix}
V_{in}
\end{bmatrix}
\]

\[
\begin{align*}
q_1 &= x_1 \\
q_2 &= \dot{x}_1 \\
q_3 &= x_2 \\
q_4 &= \dot{x}_2
\end{align*}
\]
Linear Mathematical Model:
Transfer Functions & State-Space Matrices

\[
\frac{X_1(s)}{V_{\text{in}}(s)} = \frac{3.2503s^2 + 4.5887s + 2518.7}{s^4 + 2.3449s^3 + 1265.7s^2 + 1414.1s + 284460}
\]

\[
\frac{X_2(s)}{V_{\text{in}}(s)} = \frac{1259.3}{s^4 + 2.3449s^3 + 1265.7s^2 + 1414.1s + 284460}
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-489.45 & -0.93313 & 244.72 & 0 \\
0 & 0 & 0 & 1 \\
387.45 & 0 & -774.90 & -1.4118 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
3.2503 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]
Frequency Response Plots: Analytical

\[ \frac{X_1}{V_{in}} \]

**Zero:** \(-0.706 \pm 27.8i \quad \Rightarrow \quad \omega = 27.8 \text{ rad/s} \quad \zeta = 0.0254\)

**Poles:** \(-0.536 \pm 17.1i \quad \Rightarrow \quad \omega = 17.1 \text{ rad/s} \quad \zeta = 0.0313\)

\(-0.637 \pm 31.2i \quad \Rightarrow \quad \omega = 31.2 \text{ rad/s} \quad \zeta = 0.0204\)
Frequency Response Plots: Experimental

\[ \frac{X_1}{V_{in}} \]
Frequency Response Plots: Analytical

\[ \frac{X_2}{V_{\text{in}}} \]

**Zero:** None

**Poles:**

\[-0.536 \pm 17.1i \Rightarrow \omega = 17.1 \text{ rad/s} \quad \zeta = 0.0313\]

\[-0.637 \pm 31.2i \Rightarrow \omega = 31.2 \text{ rad/s} \quad \zeta = 0.0204\]
Frequency Response Plots: Experimental

\[
\frac{X_2}{V_{in}}
\]
Compliantly-Coupled Motor and Load

\[ J_M = \text{rotor inertia of a motor} \]
\[ J_L = \text{driven-load inertia} \]
\[ K_S = \text{elasticity of coupling} \]
\[ B_{ML} = \text{viscous damping of coupling} \]
\[ B_M = \text{viscous damping between ground and motor rotor} \]
\[ B_L = \text{viscous damping between ground and load inertia} \]
\[ T = \text{electromagnetic torque applied to motor rotor} \]
• **Comments**
  
  – $K_S$, the elasticity of the coupling; it is often neglected in low-power systems; modeling it in high-power systems is essential.
  
  – $B_{ML}$, the viscous damping of the coupling; it is usually small, as transmission materials provide little damping.
  
  – $B_M$ and $B_L$ can be neglected in the following analysis, as they have a small effect on resonance. They are included here for completeness.
Coulomb friction has been neglected. The fixed value of Coulomb friction has little impact on stability when the motor is moving. At rest, the impact of stiction on resonance is more complex. Sometimes stiction is thought of as increasing the load inertia when the motor is at rest. This accounts for the tendency of systems to change resonance behavior when the motion stops.
Equations of Motion

\[ T - B_M \theta_M - B_{ML} \left( \theta_M - \theta_L \right) - K_S \left( \theta_M - \theta_L \right) = J_M \ddot{\theta}_M \]

\[-B_L \dot{\theta}_L + B_{ML} \left( \dot{\theta}_M - \dot{\theta}_L \right) + K_S \left( \theta_M - \theta_L \right) = J_L \ddot{\theta}_L \]

Laplace Transform of the Equations of Motion

\[
\begin{bmatrix}
J_M s^2 + (B_{ML} + B_M) s + K_S \\
J_L s^2 + (B_{ML} + B_L) s + K_S
\end{bmatrix}
\begin{bmatrix}
\Theta_M (s) \\
\Theta_L (s)
\end{bmatrix} = \begin{bmatrix}
(B_{ML} s + K_S) \\
(B_{ML} s + K_S)
\end{bmatrix}
\begin{bmatrix}
\Theta_L (s) \\
\Theta_M (s)
\end{bmatrix}
+ \begin{bmatrix}
T(s) \\
0
\end{bmatrix}
\]
Transfer Functions

\[
\frac{\Theta_M}{T}(s) = \frac{J_M s^2 + (B_{ML} + B_L) s + K_S}{D(s)}
\]

\[
\frac{\Theta_L}{T}(s) = \frac{B_{ML} s + K_S}{D(s)}
\]

\[
D(s) = [J_M J_L] s^4 + [(J_M + J_L) B_{ML} + J_M B_L + J_L B_M] s^3 + [(J_M + J_L) K_S + B_M B_L + B_{ML} (B_L + B_M)] s^2 + [(B_M + B_L) K_S] s
\]
MatLab / Simulink Block Diagram (BM = 0 and BL = 0)

\[
T - B_{ML} \left( \ddot{\theta}_M - \ddot{\theta}_L \right) - K_S (\theta_M - \theta_L) = J_M \dddot{\theta}_M
\]

\[
B_{ML} \left( \ddot{\theta}_M - \ddot{\theta}_L \right) + K_S (\theta_M - \theta_L) = J_L \dddot{\theta}_L
\]
Transfer Functions
(B_L = 0 and B_M = 0)

\[
\frac{\Theta_M}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{J_Ls^2 + B_{ML}s + K_S}{J_Ms^2 + B_{ML}s + K_S} \right]
\]

\[
\frac{\Theta_L}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{B_{ML}s + K_S}{J_Ms^2 + B_{ML}s + K_S} \right]
\]

As \( K_S \to \infty \),

or as \( s \to 0 \)

\[ \Theta_M(s) = \Theta_L(s) = \frac{1}{(J_M + J_L)s^2} \]

Rigid-Body Motion
Transfer Functions in Standard Form
(B_M = 0 and B_L = 0)

\[
\frac{\Theta_M}{T}(s) = K \left[ \frac{s^2}{\omega_{AR}^2} + \frac{2\zeta_{AR}s}{\omega_{AR}} + 1 \right]
\]

\[
\frac{\Theta_L}{T}(s) = \frac{K(\tau s + 1)}{s^2 \left[ \frac{s^2}{\omega_R^2} + \frac{2\zeta_R s}{\omega_R} + 1 \right]}
\]

Natural frequency of load connected to ground through the compliance

\[
K = \frac{1}{J_M + J_L}
\]

\[
\tau = \frac{B_{ML}}{K_S}
\]

\[
\omega_R = \sqrt{\frac{K_S(J_M + J_L)}{J_M J_L}}
\]

\[
\zeta_R = \frac{B_{ML}}{2 \sqrt{K_S J_M J_L}}
\]

\[
\omega_{AR} = \sqrt{\frac{K_S}{J_L}}
\]

\[
\zeta_{AR} = \frac{B_{ML}}{2 \sqrt{K_S J_L}}
\]
Compliantly-Coupled Systems

Sample Values:

\[ J_L = 0.002 \text{ kg-m}^2 \]
\[ J_M = 0.002 \text{ kg-m}^2 \]
\[ K_S = 200 \text{ N-m/rad} \]
\[ B_{ML} = 0.01 \text{ N-m-s/rad} \]

\[ \omega_{AR} = 316 \text{ rad/s} = 50.3 \text{ Hz} \]
\[ \omega_R = 447 \text{ rad/s} = 71.2 \text{ Hz} \]
\[ \zeta_{AR} = 0.008 \]
\[ \zeta_R = 0.011 \]

\[ \Theta_M(T) \]

Compliantly-Coupled Motor + Load

\[ \omega_R > \omega_{AR} \text{ always} \]

Rigidly-Coupled Motor + Load

\[ \frac{1}{J_M + J_L}s^2 \]
Limiting Behavior:

\[
\frac{\Theta_M}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{J_Ls^2 + B_{ML}s + K_S}{\frac{J_L J_M}{J_L + J_M}s^2 + B_{ML}s + K_S} \right]
\]

\[\approx \frac{1}{(J_M + J_L)s^2} \quad \text{as } \omega \to 0\]

\[\approx \frac{1}{J_M s^2} \quad \text{as } \omega \to \infty\]

\[\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}} \quad \text{As } J_L \to \infty, \quad \omega_{AR} \to 0 \quad \text{and} \quad \omega_R \to \sqrt{\frac{K_S}{J_M}}\]
\( \frac{\Theta_L}{T}(s) \)

**Note:** There is no anti-resonant frequency in this transfer function. Also, there is 90° more phase lag at high frequency.
Bode Plot Comparison

\[ \frac{\Theta_M}{T}(s) \quad \frac{\Theta_L}{T}(s) \]

90° additional phase lag
\[
\frac{\Theta_M(s)}{\Theta_L(s)} = \frac{s^2 + 2\zeta_{AR} s + 1}{\omega_{AR}^2 (\tau s + 1)}
\]
Effect of $J_L / J_M$ Ratio on Resonance and Anti-Resonance

\[
\frac{J_L}{J_M} = 1
\]

\[
\frac{J_L}{J_M} = 5
\]

\[
\frac{J_L}{J_M} = \frac{1}{5}
\]

$J_M = 0.002 \text{ kg-m}^2$

$K_S = 200 \text{ N-m/rad}$

$B_{ML} = 0.01 \text{ N-m-s/rad}$
Effect of Varying $\frac{J_L}{J_M}$

Bode Diagram

**Magnitude (dB)**

-20  -40  -60  -80  -100

5  1  0.2

**Phase (deg)**

-180  -135  -90  -45  0

10^2  10^3

Frequency (rad/sec)
\[ \frac{J_L}{J_M} = 100, 10, 1, 0.1 \]

\[ \Theta_M(s) \]

\[ \omega_R \text{ lower limit} = 316 \text{ rad/sec} \]

Bode Diagram

Magnitude (dB)

-20
-40
-60
-80
-100

317.8 rad/sec

Phase (deg)

-180
-135
-90
-45
0

Frequency (rad/sec)

\(10^2\)

\(10^3\)

Compliantly-Coupled Systems

K. Craig
\[ \frac{J_L}{J_M} = 100, \ 1000 \]

Bode Diagram

\[ \Theta_M(T(s)) \]

\[ \omega_R \text{ lower limit} = 316 \text{ rad/sec} \]
• **Observations**
  - For $J_M > 0$, the anti-resonance frequency always occurs before the resonance frequency.

  \[
  \omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}}
  \]

  \[
  \omega_{AR} = \sqrt{\frac{K_S}{J_L}}
  \]

  - At a low $J_L/J_M$ ratio, the resonance and anti-resonance frequencies are close to each other at a high frequency.
As \( J_L/J_M \) increases, both the anti-resonance and resonance frequency decrease, with the anti-resonance frequency decreasing at a faster rate.

\[
\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}} = \sqrt{\frac{1 + \frac{J_L}{J_M}}{J_L}} = \sqrt{\frac{1 + \frac{J_M}{J_L}}{J_M}}
\]

\[
\omega_{AR} = \sqrt{\frac{K_S}{J_L}} = \sqrt{\frac{1}{J_L}}
\]

At \( J_L = J_M \), \( \omega_{AR} = 0.707 \omega_R \).

As \( J_L \to \infty \), \( \omega_{AR} \to 0 \) and \( \omega_R \to \sqrt{\frac{K_S}{J_M}} \).
– For a given \( J_L \), to increase the resonance frequency, either increase the shaft stiffness or decrease the motor inertia.

\[
\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}} = \sqrt{K_S \frac{1 + \frac{J_L}{J_M}}{J_L}} = \sqrt{K_S \frac{1 + \frac{J_M}{J_L}}{J_M}}
\]

– As \( K_S \) increases, both \( \omega_R \) and \( \omega_{AR} \) increase.
– A general guideline to avoid instability problems is to keep the desired closed-loop bandwidth well below the resonance frequency and the ratio \( J_L / J_M \) less than 5.
- Heavy-Load Approximation: $J_L > 5J_M$

$$J_M + J_L \approx J_L$$

$$\frac{J_M J_L}{J_M + J_L} = \frac{J_M}{J_L} + 1 \approx J_M$$

$$\frac{\Theta_M(s)}{T} = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{J_L s^2 + B_{ML}s + K_S}{J_L J_M s^2 + B_{ML}s + K_S} \right]$$

$$\approx \frac{1}{J_L s^2} \left[ \frac{J_L s^2 + B_{ML}s + K_S}{J_M s^2 + B_{ML}s + K_S} \right]$$
Heavy-Load Approximation

\[ \frac{J_L}{J_M} = 5 \]
Example of a Control Structure: Dual-Loop Control

Note: With PD control, the inner loop acts like a low-pass filter. The I control forces the motor to move when the load accumulates a position error, thereby assuring load accuracy.
Example of a Control Structure: Dual-Loop Control

- Placing a sensor on the motor provides stable and accurate control of motor position. Transmission compliance and backlash prevent the load from reaching the exact same position.
- Placing a sensor on the load closes the loop around the compliance and backlash, but the dynamic behavior of the compliance and backlash adds delay to the loop’s response time making the system unstable.
- Dual-loop control uses two position sensors: load and motor. The motor sensor is used to close the high-frequency inner loop, and the load sensor is used for the low-frequency load-position loop.
• Without dual-loop control, the motor sensor is used to calculate the load position. This method suffers from inaccuracy because errors in transmission components distort the relationship between motor feedback and load position.

• Dual-loop control is used in applications that require high accuracy. With a load sensor, inaccuracy in the transmission components does not affect its accuracy.

• However, because the load is connected to the motor through a compliant transmission, the load sensor does not react immediately to changes in motor position; the time lag between motor rotation and load movement makes the load sensor signal too slow to close a fast loop.
• The solution is to connect the load sensor to the slow control loop where accuracy is most important and to connect the motor sensor to the fast control loop where concern for speed dominates.

• Dual-loop control corrects for inaccuracy of transmission components, but also makes the control system much less sensitive to backlash (lost motion), since the position loop is closed on actual load position.

• Dual-loop control also provides the control laws access to load position and, by calculation, the load velocity.
Belt-Driven Load and Motor

Resilient Belt

Input Torque $T$

Motor

Load

Rigid Belt Case:

$$R_1 \theta_M = R_2 \theta_L$$

$$\left[ J_M + \left( \frac{R_1}{R_2} \right)^2 J_L \right] \ddot{\theta}_M = T$$
Equations of Motion

\[
\begin{align*}
J_M \ddot{\theta}_M + B_M \dot{\theta}_M + 2KR_1^2 \theta_M + 2BR_1^2 \dot{\theta}_M &= T + 2KR_1 R_2 \theta_L + 2BR_1 R_2 \dot{\theta}_L \\
J_L \ddot{\theta}_L + B_L \dot{\theta}_L + 2KR_2^2 \theta_L + 2BR_2^2 \dot{\theta}_L &= 2KR_1 R_2 \theta_M + 2BR_1 R_2 \dot{\theta}_M
\end{align*}
\]

Laplace Transform of the Equations of Motion

\[
\begin{align*}
\left[ J_M s^2 + \left( B_M + 2BR_1^2 \right) s + 2KR_1^2 \right] \Theta_M (s) &= \left( 2BR_1 R_2 s + 2KR_1 R_2 \right) \Theta_L (s) + T(s) \\
\left[ J_L s^2 + \left( B_L + 2BR_2^2 \right) s + 2KR_2^2 \right] \Theta_L (s) &= \left( 2BR_1 R_2 s + 2KR_1 R_2 \right) \Theta_M (s) \\
\begin{bmatrix}
J_M s^2 + \left( B_M + 2BR_1^2 \right) s + 2KR_1^2 \\
-(2BR_1 R_2 s + 2KR_1 R_2)
\end{bmatrix}
\begin{bmatrix}
\Theta_M (s) \\
\Theta_L (s)
\end{bmatrix}
= 
\begin{bmatrix}
T(s) \\
0
\end{bmatrix}
\end{align*}
\]
Transfer Functions

\[
\frac{\Theta^M}{T}(s) = \frac{J_L s^2 + \left( B_L + 2BR_2^2 \right) s + 2KR_2^2}{D(s)}
\]

\[
\frac{\Theta^L}{T}(s) = \frac{2BR_1R_2 s + 2KR_1R_2}{D(s)}
\]

\[
D(s) = \left[ J_M J_L \right] s^4 + \left[ 2B \left( J_L R_1^2 + J_M R_2^2 \right) + J_M B_L + J_L B_M \right] s^3 + \left[ 2K \left( J_L R_1^2 + J_M R_2^2 \right) + B_M B_L + 2B \left( B_L R_1^2 + B_M R_2^2 \right) \right] s^2 + \left[ 2K \left( B_L R_1^2 + B_M R_2^2 \right) \right] s
\]
Transfer Functions
\((B_L = 0 \text{ and } B_M = 0)\)

\[
\Theta_m(s) = \frac{J_L s^2 + 2BR^2_2 s + 2KR^2_2}{[J_M J_L] s^4 + [2B(J_L R^2_1 + J_M R^2_2)] s^3 + [2K(J_L R^2_1 + J_M R^2_2)] s^2}
\]

\[
\Theta_L(s) = \frac{2BR_1 R_2 s + 2KR_1 R_2}{[J_M J_L] s^4 + [2B(J_L R^2_1 + J_M R^2_2)] s^3 + [2K(J_L R^2_1 + J_M R^2_2)] s^2}
\]
Transfer Functions
(BL = 0 and BM = 0)

\[
\frac{\Theta_M}{T}(s) = \frac{1}{\left( J_L R_1^2 + J_M R_2^2 \right) s^2} \left[ \frac{J_L s^2 + 2BR_2^2 s + 2KR_2^2}{\frac{J_M J_L}{J_L R_1^2 + J_M R_2^2} s^2 + 2Bs + 2K} \right]
\]

\[
\frac{\Theta_L}{T}(s) = \frac{1}{\left( J_L R_1^2 + J_M R_2^2 \right) s^2} \left[ \frac{2BR_1 R_2 s + 2KR_1 R_2}{\frac{J_M J_L}{J_L R_1^2 + J_M R_2^2} s^2 + 2Bs + 2K} \right]
\]
Transfer Functions in Standard Form
(B_M = 0 and B_L = 0)

\[
\frac{\Theta_M}{T}(s) = \frac{K_1}{s^2 \left[ \frac{s^2}{\omega_{AR}^2} + \frac{2\zeta_{AR}s}{\omega_{AR}} + 1 \right]}
\]

\[
\frac{\Theta_L}{T}(s) = \frac{K_2 (\tau s + 1)}{s^2 \left[ \frac{s^2}{\omega_{R}^2} + \frac{2\zeta_{R}s}{\omega_{R}} + 1 \right]}
\]

\[
K_2 = \frac{R_1 R_2}{J_M R_2^2 + J_L R_1^2}
\]

\[
K_1 = \frac{R_2^2}{J_M R_2^2 + J_L R_1^2}
\]

\[
\tau = \frac{B}{K}
\]

\[
\omega_R = \sqrt{\frac{2K(J_M R_2^2 + J_L R_1^2)}{J_M J_L}}
\]

\[
\zeta_R = \frac{B}{\sqrt{2KJ_M J_L}}
\]

\[
\omega_{AR} = \sqrt{\frac{2KR_2^2}{J_L}}
\]

\[
\zeta_{AR} = \frac{BR_2}{\sqrt{2KJ_L}}
\]
Classification of Resonance

- **High-Frequency Resonance**
  - Here both the resonant and anti-resonant frequencies are well above the system velocity closed-loop bandwidth.
  - This is more common on mechanically-stiff machines.
  - A small-magnitude oscillation rides on the step response.
  - The gain from the resonance degrades the gain margin. High-frequency resonance is a problem of gain margin; small changes in gain have a large impact while small changes in phase do not.
\[ E(s) = \left(\frac{\omega_{amp}^2}{s^2 + 2\zeta_{amp} \omega_{amp} s + \omega_{amp}^2}\right) \]

\[ \omega_{amp} = 5027 \text{ rad/s} \]

\[ \zeta_{amp} = 0.707 \]

Motor Velocity
Proportional Control
\[ K_P = 0.028 \]

Current Amplifier

\[ J_L = 0.0002 \text{ kg-m}^2 \]

\[ J_M = 0.0002 \text{ kg-m}^2 \]

\[ K_S = 2000 \text{ N-m/rad} \]

\[ B_{ML} = 0.01 \text{ N-m-s/rad} \]
Gain from the resonance degrades the GM

Closed-Loop Bode Plots
Bandwidth = 14.6 Hz

Open-Loop Bode Plots
GM = 27.4 dB  PM = 88.9°
• **Low-Frequency Resonance**
  
  – Here the anti-resonant frequency is near the closed-loop velocity bandwidth.
  
  – This more common on more compliant machines.
  
  – In the open-loop frequency-response plot, there are 3 crossover frequencies; the highest frequency crossover causes instability, where the phase is near -180°.
  
  – Low-frequency resonance is a problem of phase margin; small changes in phase have a large impact, whereas small changes in gain do not.
\[
\frac{E(s)}{I(s)} = \frac{\omega_{\text{amp}}^2}{s^2 + 2\zeta_{\text{amp}} \omega_{\text{amp}} s + \omega_{\text{amp}}^2}
\]

\[\omega_{\text{amp}} = 5027 \text{ rad/s}\]

\[\zeta_{\text{amp}} = 0.707\]

Current Amplifier

Motor Velocity
Proportional Control

\[K_P = 0.28\]

\[J_L = 0.0002 \text{ kg-m}^2\]

\[J_M = 0.0002 \text{ kg-m}^2\]

\[K_S = 2000 \text{ N-m/rad}\]

\[B_{ML} = 0.01 \text{ N-m-s/rad}\]
Compliantly-Coupled Systems

Closed-Loop Bode Plots
Bandwidth = 135 Hz

Open-Loop Bode Plots
GM = 7.4 dB  PM = 13.1°

Phase from the resonance degrades the PM
Curing Resonance: Mechanical Cures

- **Stiffen the Transmission**
  - This usually improves resonance problems.
  - The key to stiffening a transmission is to improve the loosest components in the transmission in an effort to raise the total spring constant $K_S$. This has the effect of raising both the resonant and anti-resonant frequencies and moving them away from the frequencies where they cause harm.
  - Some Suggestions:
    - Use multiple belts, wide belts, or reinforce belts
    - Shorten shafts; use large-diameter shafts

\[ K = \frac{JG}{\ell} = \frac{\pi d^4}{32\ell} \]
• Use stiffer gear boxes
• Use larger lead screws and stiffer ball nuts
• Use idlers to support belts that run long distances
• Reinforce the frame of a machine
• Oversize coupling components; be cautious here as this will also add inertia, which may slow peak acceleration
  – When stiffening a machine, start with the loosest components, as a single loose component can single-handedly reduce the overall spring constant significantly.
• **Add Damping**
  – In practice, it is difficult to add damping between the motor and load. Materials with large inherent damping do not normally make good transmission components.

\[
K_{\text{total}} = \frac{1}{K_{\text{coupling}}} + \frac{1}{K_{\text{gear box}}} + \ldots + \frac{1}{K_n}
\]
– Steps used to stiffen a machine can actually make the machine perform more poorly because they also reduce damping.
– Sometimes the unexpected loss of damping can cause resonance problems.

• **Reduce Load-to-Motor Inertia Ratio**
  – Reducing the load-to-motor inertia ratio will improve resonance problems. The smaller the \( J_L / J_M \) ratio, the less compliance will affect the system.
  – At low frequency, the system appears to have a noncompliant inertia, \( J_T = J_L + J_M \).
  – At high frequency, the load inertia is disconnected; the system sees only \( J_M \), the motor inertia.
Compliantly-Coupled Motor + Load

Rigidly-Coupled Motor + Load

Compliantly-Coupled Systems

\[ \omega_R > \omega_{AR} \text{ always} \]
– In a sense, compliance gives the system an apparent inertia that varies with frequency. The smaller the $J_L/J_M$ ratio, the less variation in apparent inertia as would be indicated by a smaller distance between the two parallel lines representing the low-frequency and high-frequency amplitude ratio vs. frequency plots.

– Reducing the load inertia is the best way to reduce the ratio $J_L/J_M$; reduce the mass of the load or change its dimensions.

– The reflected inertia (load inertia felt by the motor) can also be reduced by increasing the gear ratio.

\[
\theta_M = N \theta_L \quad J_{\text{effective}} = J_M + \frac{1}{N^2} J_L
\]
– Unfortunately, increasing N can reduce the top speed of the application. Similar effects are realized by changing lead screw pitch or pulley diameter ratios.

– Any steps taken to reduce the load inertia will usually help the resonance problem; most machine designers work hard to minimize load inertia for non-servo reasons, e.g., cost, peak acceleration, weight, structural stress.

– Increasing $J_M$ does help the resonance problem. Unfortunately, raising motor inertia increases the total inertia, which reduces total acceleration or requires more torque and power from the drive to maintain the acceleration. Increasing motor inertia therefore increases the cost of both motor and drive. Despite the increase in size and cost by increasing $J_M$, it is commonly used because it so effectively improves resonance problems.
– **Common Misconception:** $J_L/J_M$ ratio is optimized when the ratio is 1 or inertias are equal or matched.

– Based on a fixed $J_M$ and $J_L$, the gear ratio $N$ that maximizes the power transferred from motor to load is the ratio that forces the reflected load inertia $J_L/N^2$ to be equal to the rotor inertia $J_M$. This fact has little bearing on how motors and gear ratios are selected in practice because the assumption that the motor inertia is fixed is usually invalid; each time the gear ratio increases, the required torque from the motor decreases, allowing the use of a smaller motor.

– The primary reason that $J_M$ and $J_L$ should be matched is to reduce resonance problems; actually, this is an oversimplification.
Larger $J_M$ improves resonance but increases cost. The more responsive the control system, the smaller the $J_L/J_M$ ratio. $J_L/J_M$ ratios of 3 to 5 are common in typical servo applications. Highest bandwidth applications require that the load inertia be no larger than about 70% of the motor inertia. The $J_L/J_M$ ratio also depends on the compliance of the machine; stiffer machines will forgive large load inertias.
Let’s focus on improving the performance of a servo system in the presence of a low-frequency (100 Hz) lightly-damped resonance, as this type of resonance occurs most often in general industrial machines.

Command response and dynamic stiffness (disturbance rejection) are two key performance ratings for high-performance applications, and mechanical resonance is one of the most common obstacles to achieving high performance.

The common situation in industry is to have a single motion sensor on the motor.
• PI and PID controllers are typically used to control the lumped-inertia, rigid-body plant, but the dual quadratic (bi-quad) term alters the phase and gain of the rigid-body plant.

\[ \frac{\dot{\theta}_M}{T}(s) = \begin{bmatrix} \frac{1}{(J_M + J_L)s} \\ \frac{J_Ls^2 + B_{ML}s + K_s}{J_LJ_Ms^2 + B_{ML}s + K_s} \end{bmatrix} \]

- Rigid-Body Term
- Dual-Quadratic Term

• The resonance and anti-resonance are usually both lightly damped.
• Methods of controlling resonance rely on modifying the effects of the bi-quad term.
• **Common Control Structure**

![Diagram of Common Control Structure](image-url)
The most common anti-resonance filters are based on one of three filtering techniques: low-pass, notch, and bi-quad filters.

The motor position is read from an encoder or other motor position sensor and used to calculate an average velocity feedback signal.

Two observers are shown, both fed by electromagnetic torque and motor position.

- **Rigid-Body Observer** which outputs motor velocity and acceleration assuming no knowledge of the load inertia or coupling stiffness.
- **Compliant-Body Observer** which models the motor and load velocities and motor acceleration based on knowledge of the load and coupling characteristics.

The goal is to provide increased dynamic stiffness, faster command response, and insensitivity to mechanical changes.
**Evaluation System:**
- Motor driving a load through a compliant coupling
- Servo amplifier operating in the current mode
- Motor parameters: $J_M = 0.0014 \text{ kg-m}^2$, $K_T = 0.985 \text{ N-m/A}$
- $J_L = 0.0028 \text{ kg-m}^2$
- Coupling compliance: $K_S = 372 \text{ N-m/rad}$
- $B_{ML} = 0.008 \text{ N-m-s/rad}$
- $J_L/J_M = 2$
- $B_M = B_L = 0$
\( \omega_R = 100.5 \text{ Hz} \)
\( \omega_{AR} = 58.0 \text{ Hz} \)
\( \zeta_R = 0.0068 \)
\( \zeta_{AR} = 0.0039 \)

Key problem in low-frequency resonance is the increase in gain at frequencies above \( \omega_R \).

\( \Theta_M \) \( \frac{T}{s} \)

Compliantly-Coupled Motor + Load

Rigidly-Coupled Motor + Load

\[ \frac{1}{(J_M + J_L)s^2} \]
\[ \Theta_L(s) \]
$\frac{\Theta_M}{T}(s)$
\[ \frac{\Theta_L}{T}(s) \]
• **Low-Pass Filter** \( \frac{K}{\tau s + 1} \)

• This is the most common method used to control resonance. The filter is placed in the loop to compensate for the change in gain presented by the compliant load. It is used to reduce the gain at the resonant frequency. This improves the gain margin at or near the resonant frequency. However, it also degrades the phase margin.

• The key advantage is that it is easy to use. Only the break frequency needs to be adjusted for the given load resonance.

• The most common position for the filter is just before the servo amplifier.
• Low-pass filters work well for high-frequency resonance. High-frequency resonance is a problem of gain margin; resonance raises the gain at high frequency, eroding the gain margin. Filters reduce the gain at high frequency, restoring gain margin.

• Unfortunately, the low-pass filter is not very effective for the commonly-found case of low-frequency resonance, where the resonant frequency is no higher than 5 or 10 times the velocity-loop bandwidth.

• Low-frequency resonance is a problem of phase margin. Using filters exacerbate problems with phase margin. In fact, attempting to use the filter with the low-frequency resonance problem will usually increase instability.
Low-Pass Filter

\[
\frac{K}{\tau s + 1}
\]

\[K = 1\]

\[\tau = 0.0016\]

bandwidth \(\frac{1}{\tau} = 628 \text{ rad/s}\)

\(= 100 \text{ Hz}\)

-3 dB

-45°
• **Notch Filters**

\[
\frac{s^2 + \omega_N^2}{s^2 + 2\zeta_N\omega_N s + \omega_N^2}
\]

• The primary problem of low-pass filters is that they introduce phase lag in the loop, reducing the phase margin at the gain crossover.

• Notch filters add little phase lag below the notch frequency.

• Like low-pass filters, notch filters work well for high-frequency resonance.

• The problem with notch filters is that their benefits are limited to a narrow frequency range. If the resonant frequency changes, the notch filter will not be effective. Unfortunately, resonant frequencies commonly change on practical machines, e.g., load inertia changes, compliance changes.
• Notch filters work best when the machine construction and operation allow variation of the resonant frequency and when the notch can be individually tuned for each machine.
• Notch filters do not work well on low-frequency resonance because of the effect of the anti-resonance frequency, which is near the passband of the velocity loop.
• Usually $\omega_N$ is selected to approximate the resonant frequency, $\omega_R$, and the damping ratio, $\zeta_N$, is selected to be moderate to low, below 0.4. The notch filter, like the low-pass filter, is used to increase gain margin by attenuating the open-loop gain in the frequency region near the resonant frequency. The damping ratio is generally selected such that less phase distortion occurs at lower frequencies than when using a low-pass filter. This allows higher gains in the control loop. The notch filter also passes the frequencies above the resonant frequency.
**Notch Filter**

\[
\frac{s^2 + \omega_N^2}{s^2 + 2\zeta_N \omega_N s + \omega_N^2}
\]

\(\omega_N = 591 \text{ rad/s}\)

\(\zeta_N = 0.4\)
• **Bi-Quad Filter**

\[
\left( \frac{s^2 + 2\zeta_z \omega_z s + \omega_z^2}{s^2 + 2\zeta_p \omega_p s + \omega_p^2} \right) \left( \frac{\omega_p^2}{\omega_n^2} \right)
\]

• The notch filter can be expanded to a full bi-quad filter which will work better with low-frequency resonance.

• A bi-quad filter is a notch filter where the damping and natural frequencies of the numerator and denominator can be set independently. The constant term adjusts the DC gain for unity.

• By tuning the bi-quad term, stability of motor control can be restored. Unfortunately, the load position is not well controlled in these cases. The load must oscillate to help smooth the motor position.
The filter is quite sensitive to both load inertia and compliance, as it must be tuned to both resonant and anti-resonant frequencies.

If complete cancellation is achieved, the effect of the bi-quad filter is to eliminate the bi-quad term in the motor position transfer function:

\[
\frac{\Theta_M}{T}(s) = K \left[ \frac{s^2}{\omega_{AR}^2} + \frac{2\zeta_{AR}s}{\omega_{AR}} + 1 \right] \left[ \frac{s^2}{\omega_{RR}^2} + \frac{2\zeta_{RR}s}{\omega_{RR}} + 1 \right]
\]

The command response and dynamic stiffness can be enhanced considerably.

The two shortcomings, load oscillation and sensitivity to parameter changes, are major ones, however.
Bi-Quad Filter

\[
\frac{s^2 + 2\zeta_z \omega_z s + \omega_z^2}{s^2 + 2\zeta_p \omega_p s + \omega_p^2} \left( \frac{\omega_p^2}{\omega_n^2} \right)
\]

\[
\begin{align*}
\omega_z &= 100.5 \text{ Hz} \\
\omega_p &= 58.0 \text{ Hz} \\
\zeta_z &= 0.0068 \\
\zeta_p &= 0.0039
\end{align*}
\]
Curing Resonance: Observer-Based Techniques

- These anti-resonance methods require one or more signals in addition to the motor-mounted encoder. Most applications cannot afford additional sensors because of the cost of purchasing and mounting sensors, and the loss of reliability from extra components and wiring.
- There are two observers that can be used to obtain signals in addition to the encoder:
  - Rigid-Body Observer
  - Compliant-Body Observer
• **Rigid-Body Observer**

  The rigid-body observer provides observed motor velocity and acceleration. Its structure is directly analogous to the motor. The electromagnetic torque $T$ is a feedforward input to the observer. Observed acceleration, velocity, and position are generated. The PID control law drives the observer error to zero.

$$K_D \left( s + K_P + \frac{K_I}{s} \right)$$

![Diagram of the Rigid-Body Observer](image)
• The PID control gains can be set as equivalent to a three-pole Butterworth (maximally flat) filter:

\[ K_D = 2\omega_{AR}, \quad K_P = \omega_{AR}, \quad K_I = \frac{\omega_{AR}^2}{2} \]

• From the block diagram, we see that:

\[
\hat{V}_M = \left[ \frac{T}{J_ms} \right] \left[ \frac{s^3}{s^3 + K_D \left( s^2 + K_PS + K_I \right)} \right] + \left[ \mathbf{P}_M \right] \left[ \frac{K_D \left( s^2 + K_PS + K_I \right)}{s^3 + K_D \left( s^2 + K_PS + K_I \right)} \right]
\]

• This represents a high-pass filtered velocity signal from the torque and a low-pass filtered velocity signal from the encoder. The observer removes high frequency content of the encoder signal to reduce resonance problems. It also fills in missing frequency content with information from T which enables it to produce zero phase lag at lower frequencies and higher frequencies.
• **Compliant-Body Observer**

• An alternative to the rigid-body observer is to use a model that includes the motor, load, and the coupling.
• The principles of the compliant-body observer are similar to those of the rigid-body observer. The observer can observe both motor and load acceleration, velocity, and position.

• **Observer Error**

• An observer can be evaluated by how well it predicts the motor velocity in the presence of a torque disturbance.

• Normally the error is low at low frequency because of the integral term in the PID controller. It is also low at high frequency because the inertia reduces the effects of the disturbance at high frequency.

• The error will be largest in the mid-range frequencies.
• **Observer-Based, Anti-Resonance Methods**

• The methods discussed here use one or more observed signals in addition to the motor position feedback.

• **Acceleration Feedback**

• Acceleration feedback effectively increases the motor inertia. Its conceptual implementation is shown below.

$$K_A s^2$$

$$\frac{1}{(J_M + J_L)} s^2$$

$$\frac{J_L s^2 + B_{ML} s + K_S}{\frac{J_L J_M}{J_L + J_M} s^2 + B_{ML} s + K_S}$$
• By combining the forward and feedback paths, it can be seen that the effective motor inertia increases to \((1+K_A)J_M\). Increasing physical motor inertia is a well-known method of reducing sensitivity to mechanical resonance. Acceleration feedback produces similar benefits without the negatives of physical motor mass such as increased weight and size and reduced peak acceleration.

• Acceleration feedback is implemented by adding to the current command a term proportional to observed acceleration:

\[
I_C = \frac{K_A \hat{A}_M J_M}{K_T}
\]

• \(I_C\) is the current command; \(K_T\) is the motor torque constant.
• Quantization noise in the observed acceleration and phase lag in the current regulator limit how much acceleration feedback is possible. $K_A$ is typically $< 2.5$.
• An alternative to using observed acceleration feedback is to measure average acceleration by double differentiating position. This is inferior to the observer because differentiation maximizes quantization noise and induces phase lag.
- **Observer Filtering**
  - Observer filtering here refers to the use of the observer to filter high-frequency signals from the encoder, filling in the missing information with the electromagnetic torque signal.
  - This is superior to ordinary low-pass filters since it should yield zero phase lag in the observed velocity at lower frequencies and thus has less effect on loop stability.

- **Active Resonance Damping**
  - Active resonance damping adds a torque in proportion to the difference of motor and load observed speeds:

\[
T_C = B_{ML} \left( \hat{V}_M - \hat{V}_L \right)
\]
• Active damping increases the effective physical damping similar to the way acceleration feedback increases effective inertia.

• Active damping is well known to cure resonance when a physical sensor is placed on the load. The question here is whether the compliant-body observer can be used to provide load and motor velocity allowing use of active damping without an additional sensor.
• **Center-of-Mass Control**

  Center-of-mass control used motor and load position sensors in combination to eliminate resonance. The method uses velocity feedback as a combination of motor and load velocity weighted by their inertias:

  \[
  V_F = V_L \frac{J_L}{J_L + J_M} + V_M \frac{J_M}{J_L + J_M}
  \]

  • \(V_F\) provides the velocity of neither the motor nor the load, but the center of mass of the motor and load. While the motor and load may ring, control-loop instability is eliminated because the control loop sees only the center of mass; the inertia of the center of mass does not vary with frequency.
• Using $V_F$ reduces sensitivity to resonance between motor and load because during such resonance, the center-of-mass doesn’t move.
• If the load is heavy, the oscillations of the load are small, although the motor may ring quite a bit. Load oscillation is much more noticeable with low-frequency resonance.
• The major shortcoming of the center-of-mass method is that a feedback sensor must be placed on the load. However, the compliant-body observer can provide motor and load velocity.