Digital Implementation of Control

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Digital Implementation of Control

Anti-Aliasing Filter

A/D Converter

Sampling System

Digital Computer

D/A Converter

Final Control Element

Sensor

Plant

Sampled & Quantized Measurement

Digital Set Point

Sampled & Quantized Control Signal

Digital Control of Dynamic Systems
Advantages of Digital Control

• The current trend toward using dedicated, microprocessor-based, and often decentralized (distributed) digital control systems in industrial applications can be rationalized in terms of the major advantages of digital control:
  – Digital control is less susceptible to noise or parameter variation in instrumentation because data can be represented, generated, transmitted, and processed as binary words, with bits possessing two identifiable states.
– Very high accuracy and speed are possible through digital processing. Hardware implementation is usually faster than software implementation.

– Digital control can handle repetitive tasks extremely well, through programming.

– Complex control laws and signal conditioning methods that might be impractical to implement using analog devices can be programmed.

– High reliability can be achieved by minimizing analog hardware components and through decentralization using dedicated microprocessors for various control tasks.
- Large amounts of data can be stored using compact, high-density data storage methods.
- Data can be stored or maintained for very long periods of time without drift and without being affected by adverse environmental conditions.
- Fast data transmission is possible over long distances without introducing dynamic delays, as in analog systems.
- Digital control has easy and fast data retrieval capabilities.
- Digital processing uses low operational voltages (e.g., 0 - 12 V DC).
- Digital control has low overall cost.
Digital Signals are:
- discrete in time
- quantized in amplitude

You must understand the effects of:
- sample period
- quantization size

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• In a real sense, the problems of analysis and design of digital control systems are concerned with taking into account the effects of the sampling period, $T$, and the quantization size, $q$.

• If both $T$ and $q$ are extremely small (i.e., sampling frequency 50 or more times the system bandwidth with a 16-bit word size), digital signals are nearly continuous, and continuous methods of analysis and design can be used.

• It is most important to understand the effects of all sample rates, fast and slow, and the effects of quantization for large and small word sizes.

• It is worthy to note that the single most important impact of implementing a control system digitally is the delay associated with the D/A converter, i.e., $T/2$. 
Simulation of Continuous and Quantized Signal
Continuous Output and D/A Output
• The analog feedback signal coming from the sensor contains useful information related to controllable disturbances (relatively low frequency), but also may often include higher frequency "noise" due to uncontrollable disturbances (too fast for control system correction), measurement noise, and stray electrical pickup. Such noise signals cause difficulties in analog systems and low-pass filtering is often needed to allow good control performance.
In digital systems, a phenomenon called \textit{aliasing} introduces some new aspects to the area of noise problems. If a signal containing high frequencies is sampled too infrequently, the output signal of the sampler contains low-frequency ("aliased") components not present in the signal before sampling. If we base our control actions on these false low-frequency components, they will, of course, result in poor control. The theoretical absolute minimum sampling rate to prevent aliasing is 2 samples per cycle; however, in practice, rates of about 10 are more commonly used. A high-frequency signal, inadequately sampled, can produce a reconstructed function of a much lower frequency, which can not be distinguished from that produced by adequate sampling of a low-frequency function.
Simulation of Continuous and Sampled Signal
• Shown is an example of a computer-controlled motion control system.

\[ M = 0.001295 \text{ lbf-s}^2/\text{in} \]
\[ B = 0.259 \text{ lbf-s/in} \]
\[ K_{mf} = 0.5 \text{ lbf/A} \]
\[ K_{pa} = 2.0 \text{ A/V} \]
Sensor Gain = 1.0 V/in
Computational Delay = 0.008 s

pure/ideal mass and damper
Block Diagram of a Digital Controller
Digital Implementation of Continuous Controllers

- A digital controller differs from an analog controller in that the signals must be sampled and quantized.
- A signal to be used in digital logic needs to be sampled first; then the samples need to be converted by an analog-to-digital (A/D) converter into a quantized digital number.
- Once the digital computer has calculated the proper next control signal value, this value needs to be converted into a voltage and held constant or otherwise extrapolated by a digital-to-analog converted (D/A) in order to be applied to the actuator of the process.
- The control signal is not changed until the next sampling period.
• As a result of the sampling, there are more strict limits on the speed or bandwidth of a digital controller than on analog devices.

• A reasonable rule of thumb for selecting the sampling period is that during the rise time of the response to a step, the input to the discrete controller should be sampled approximately 6 times. By adjusting the controller for the effects of sampling, the sampling can be adjusted to 2 to 3 times per rise time. This corresponds to a sampling frequency that is 10 to 20 times the system’s closed-loop bandwidth.

• The quantization of the controller signals introduces an equivalent extra noise into the system, and to keep this interference at an acceptable level, the A/D converter usually has an accuracy of 10 to 12 bits.
• We will consider a simplified technique for finding a discrete (sampled, but not quantized) equivalent to a given continuous controller.
• The method depends on the sampling period $T_s$ being short enough that the reconstructed control signal is close to the signal that the original analog controller would have produced.
• We also assume that the numbers used in the digital logic have enough accurate bits so that the quantization implied in the A/D and D/A processes can be ignored.
• Finding a discrete equivalent to a given analog controller is equivalent to finding a recurrence equation for the samples of the control which will approximate the differential equation of the controller.
The assumption is that we have the transfer function of an analog controller and wish to replace it with a discrete controller that will accept samples of the controller input $e(kT_s)$, from a sampler and, using past values of the control signal, $u(kT_s)$, and present and past values of the input, $e(kT_s)$, will compute the next control signal to be sent to the actuator.

Let’s consider the PID controller, as an example. The proportional-integral-derivative (PID) controller is the most widely used controller in use today. It can stabilize a system, increase the speed of response of a system, and reduce steady-state errors of a system.

\[
\begin{align*}
  u(t) &= K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt} \\
  U(s) &= \left( K_p + \frac{K_i}{s} + K_D s \right) E(s)
\end{align*}
\]
• **Proportional Control**
  
  – Virtually all controllers have a large proportional gain. While we will see that derivative gain can provide incremental improvements at high frequencies, and integral gain improves performance at lower frequencies, the proportional gain is the primary actor across the entire frequency range of operation.
  
  – Here the manipulating variable U is directly proportional to the actuating signal E.
  
  – The corrective effort is made proportional to system "error"; large errors engender a stronger response than do small ones. We can vary in a continuous fashion the energy and/or material sent to the controlled process.
Proportional control exhibits nonzero steady-state errors for even the least-demanding commands and disturbances.

• Why is this so? Suppose for an initial equilibrium operating point $x_c = x_v$ and steady-state error is zero. Now ask $x_c$ to go to a new value $x_{vs}$. It takes a different value for the manipulated input $U$ to reach equilibrium at the new $x_c$. When the manipulated input $U$ is proportional to the actuating signal $E$, a new $U$ can only be achieved if $E$ is different from zero which requires $x_c \neq x_v$; thus, there must be a steady-state error.

$$u_p(t) = K_p e(t)$$

$$u_p(kT_s + T_s) = K_p e(kT_s + T_s)$$
• **Integral Control**
  
  – When a proportional controller can use large loop gain and preserve good relative stability, system performance, including those on steady-state error, may often be met.
  
  – However, if difficult process dynamics such as significant dead times prevent use of large gains, steady-state error performance may be unacceptable.
  
  – When human process operators notice the existence of steady-state errors due to changes in desired value and/or disturbance they can correct for these by changing the desired value ("set point") or the controller output bias until the error disappears. This is called *manual reset*.
Integral control is a means of removing steady-state errors without the need for manual reset. It is sometimes called *automatic reset*.

\[ \frac{du(t)}{dt} = K_1 e(t) \]

\[ \frac{U(s)}{E(s)} = \frac{K_1}{s} \]

\[ u(t) = K_1 \int_0^t e(\tau) d\tau \]

- If the value of \( e(t) \) is doubled, then the value of \( u(t) \) varies twice as fast.
- For \( e(t) = 0 \), \( u(t) \) remains stationary.
- We have seen why proportional control suffers from steady-state errors. We need a control that can provide any needed steady output (within its design range, of course) when its input (system error) is zero.
**Comparison**: Proportional vs. Integral Control

Integral control has the undesirable side effects of reducing response speed and degrading stability.
– Although integral control is very useful for removing or reducing steady-state errors, it has the undesirable side effect of reducing response speed and degrading stability.

– Why? Reduction in speed is most readily seen in the time domain, where a step input (a sudden change) to an integrator causes a ramp output, a much more gradual change.

– Stability degradation is most apparent in the frequency domain (Nyquist Criterion) where the integrator reduces the phase margin by giving an additional 90 degrees of phase lag at every frequency, rotating the \((B/E)(i\omega)\) curve toward the unstable region near the -1 point.
– Occasionally an integrating effect will naturally appear in a system element (actuator, process, etc.) other than the controller.

– These gratuitous integrators can be effective in reducing steady-state errors. Although controllers with a single integrator are most common, double (and occasionally triple) integrators are useful for the more difficult steady-state error problems, although they require careful stability augmentation.

– Conventionally, the number of integrators between E and C in the forward path has been called the *system type number*. 
In addition to the number of integrators, their location (relative to disturbance injection points) determines their effectiveness in removing steady-state errors.

In Figure (a) the integrator gives zero steady-state error for a step command but not for a step disturbance.

By relocating the integrator as in Figure (b), either or both step inputs $V_s$ and $U_s$ can be "canceled" by $M$ without requiring $E$ to be nonzero.

Integrators must be located upstream from disturbance-injection points if they are to be effective in removing steady-state errors due to disturbances.

Location is not significant for steady-state errors caused by commands.
– Integral control can be used by itself or in combination with other control modes. Proportional + Integral (PI) Control is the most common mode.

– Integral gain provides DC and low-frequency stiffness. When a DC error occurs, the integral gain will move to correct it. The higher the gain, the faster the correction. Fast correction implies a stiffer system.

– Don’t confuse DC stiffness with dynamic stiffness. A system can be quite stiff at DC and not stiff at all at high frequencies! Higher integral gains will provide higher DC stiffness but will not substantially improve stiffness at or above the loop bandwidth.
– PI controllers are more complicated to implement than P controllers. Saturation becomes more complicated, as integral wind-up must be avoided. In analog controllers, clamping diodes must be added, and in digital controllers, saturation algorithms must be coded.

– Integral gain can cause instability. In the open loop, the integral, with its 90° phase lag, reduces PM. In the time domain, the common result of adding integral gain is overshoot and ringing. As a result, larger integral gains usually reduce bandwidth.
\[
u_I(kT_s + T_s) = K_I \int_0^{kT_s + T_s} e(\tau) d\tau \\
= K_I \int_0^{kT_s} e(\tau) d\tau + K_I \int_{kT_s}^{kT_s + T_s} e(\tau) d\tau \\
= u_I(kT_s) + K_I [\text{area under } e(\tau) \text{ over one period}] \\
\approx u_I(kT_s) + K_I \frac{T_s}{2} \left[ e(kT_s + T_s) + e(kT_s) \right]
\]

Graphical Interpretation of Numerical Integration:
Area of Trapezoid
• **Derivative Control**
  
  – Proportional and integral control actions can be used as the sole effect in a practical controller.
  
  – But the various derivative control modes are always used in combination with some more basic control law. This is because the derivative mode produces no corrective effect for any constant error, no matter how large, and therefore would allow uncontrolled steady-state errors.
  
  – One of the most important contributions of derivative control is in system stability augmentation. If absolute or relative stability is the problem, a suitable derivative control mode is often the answer.
  
  – The stabilization or "damping" aspect can easily be understood qualitatively from the following discussion.
Invention of integral control may have been stimulated by the human process operators’ desire to automate their task of manual reset. Derivative control hardware may first have been devised as a mimicking of human response to changing error signals. Suppose a human process operator is given a display of system error $E$ and has the task of changing manipulated variable $M$ (say with a control dial) so as to keep $E$ close to zero.
– If you were the operator, would you produce the same value of M at $t_1$ as at $t_2$? A proportional controller would do exactly that.

– A stronger corrective effect seems appropriate at $t_1$ and a lesser one at $t_2$ since at $t_1$ the error $E$ is $E_{1,2}$ and increasing, whereas at $t_2$ it is also $E_{1,2}$ but decreasing.

– The human eye and brain senses not only the ordinate of the curve but also its trend or slope. Slope is clearly $dE/dt$, so to mechanize this desirable human response we need a controller sensitive to error derivative.

– Such a control can, however, not be used alone since it does not oppose steady errors of any size, as at $t_3$, thus a combination of proportional + derivative control, for example, makes sense.
The relation of the general concept of derivative control to the specific effect of viscous damping in mechanical systems can be appreciated from the figure below.

Here an applied torque $T$ tries to control position $\theta$ of an inertia $J$. The damper torque on $J$ behaves exactly like a derivative control mode in that it always opposes velocity $d\theta/dt$ with a strength proportional to $d\theta/dt$ making motion less oscillatory.
Derivatives of $E$, $C$, and almost any available signal in the system are candidates for a useful derivative control mode.

First derivatives are most common and easiest to implement.

The noise-accentuating characteristics of derivative operations may often require use of approximate (low-pass filtered) derivative signals.

Derivative signals can sometimes be realized better with sensors directly responsive to the desired value, rather than trying to differentiate an available signal.

In addition to stability augmentation, derivative modes may also offer improvements in speed of response and steady-state errors.
– The derivative gain advances the phase of the loop by virtue of the $90^\circ$ phase lead of a derivative. Using derivative gain will usually allow the system responsiveness to increase, allowing the bandwidth to nearly double in some cases.

– Derivative gain has high gain at high frequencies. So while some derivative gain does help the phase margin, too much hurts the gain margin by adding gain at the phase crossover frequency, typically a high frequency. This makes the derivative gain difficult to tune. The designer sees overshoot improve because of increased PM, but a high-frequency oscillation, which comes from reduced GM, becomes apparent.
– Derivatives are also very sensitive to noise. The derivative gain needs to followed by a low-pass filter to reduce noise content. However, the lower break frequency of the filter, the less benefit can be gained from the derivative gain.

– **Proportional + Derivative Control**

\[ u(t) = K_P e(t) + K_D \frac{de(t)}{dt} \]

\[ \frac{U(s)}{E(s)} = K_P + K_D s \]

– Derivative control has an anticipatory character, however, it can never anticipate any action that has not yet taken place.

– Derivative control amplifies noise signals and may cause a saturation effect in the actuator.
In the derivative term, the roles of $u$ and $e$ are reversed from integration and a consistent approximation can be written down at once.

\[ u_I(kT_s + T_s) \approx u_I(kT_s) + K_I \frac{T_s}{2} \left[ e(kT_s + T_s) + e(kT_s) \right] \]

**Integration**

\[ \frac{T_s}{2} \left[ u_D(kT_s + T_s) + u_D(kT_s) \right] = K_D \left[ e(kT_s + T_s) - e(kT_s) \right] \]

**Differentiation**
• **Z Operator**
  - The Laplace Transform variable $s$ is a differential operator. The Z Transform variable $z$ is a prediction operator or a forward-shift operator.

$$U(z) \text{ is the transform of } u(kT_s)$$

$$zU(z) \text{ is the transform of } u(kT_s + T_s)$$

- Consider the integral term.

$$u_I(kT_s + T_s) \approx u_I(kT_s) + K_I \frac{T_s}{2} \left[ e(kT_s + T_s) + e(kT_s) \right]$$

$$zU_I(z) = U_I(z) + K_I \frac{T_s}{2} \left[ zE(z) + E(z) \right]$$

$$U_I(z) = K_I \frac{T_s}{2} \frac{z+1}{z-1} E(z)$$
– The derivative term is the inverse of the integral term.

\[ U_D(z) = K_D \frac{2}{T_s} \frac{z - 1}{z + 1} E(z) \]

– The complete discrete PID controller is thus described by:

\[ U(z) = \left( K_p + K_I \frac{T_s}{2} \frac{z + 1}{z - 1} + K_D \frac{2}{T_s} \frac{z - 1}{z + 1} \right) E(z) \]

– The effect of the discrete approximation in the z-domain is as if everywhere in the analog transfer function the operator \( s \) has been replaced by the composite operator \( \frac{2}{T_s} \frac{z - 1}{z + 1} \)

– The discrete equivalent to \( D_a(s) \) is

\[ D_d(z) = D_a \left( \frac{2}{T_s} \frac{z - 1}{z + 1} \right) \]
**Example Problem**

\[
G(s) = \frac{Y}{U} = \frac{45}{(s + 9)(s + 5)} \quad \text{Plant Transfer Function}
\]

\[
D(s) = \frac{U}{E} = 1.4 \frac{s + 6}{s} \quad \text{PI Controller Transfer Function}
\]

- The closed-loop system has a rise time of about 0.2 seconds and an overshoot of about 20%.
- What is the discrete equivalent of this controller? Compare the step responses and control signals of the two systems. Consider a sample period of 0.07 seconds (about three samples per rise time) and a sample period of 0.035 seconds (about 6 samples per rise time).
\[ D_d(z) = 1.4 \frac{1.21z - 0.79}{z - 1} \quad \text{for } T_s = 0.07 \]

\[ u(k+1) = u(k) + 1.4 \left[ 1.21e(k+1) - 0.79e(k) \right] \]

\[ D_d(z) = 1.4 \frac{1.105z - 0.895}{z - 1} \quad \text{for } T_s = 0.035 \]

\[ u(k+1) = u(k) + 1.4 \left[ 1.105e(k+1) - 0.895e(k) \right] \]