Control of a Liquid-Level Process

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Valve Positioner \( p \) \( x_v \)

Electro-Pneumatic Transducer \( e_A \)

Amplifier \( e_E \) \( e_R \)

Potentiometer Bridge \( e_B \)

K

K

K

Q_M \( h_v \)

TANK

PUMP

P_U \( R_f \)

Float Level Sensor \( h_c \)
• **Objective**
  – Maintain tank level $h_C$ at the desired level $h_V$ in the face of disturbances pressure $P_U(t)$ (psig) and volume flow rate $Q_U(t)$ (ft$^3$/sec). $R_f$ is a linearized flow resistance with units psi/(ft$^3$/sec).

• **Equilibrium Operating Point**
  – All variables are steady
  – Inflow $Q_M$ exactly matches the two outflows
  – $h_C = h_V$ and $e_E = 0$
    • When $e_E = 0$, $Q_M$ can be nonzero since the electropneumatic transducer has a zero adjustment and the valve positioner has a zero adjustment, e.g., $p = 9$ psig and the valve opening corresponds to equilibrium flow $Q_M$.
    • We will deal with small perturbations in all variables away from the initial steady state.
• **Assumptions and Equations of Motion**
  
• **Tank Process Dynamics**
  
  – Density of fluid $\rho$ is constant.

\[
Q_M - Q_U - \frac{\rho gh_c - P_U}{R_f} = A_T \frac{dh_c}{dt} \\
(\tau_p s + 1) h_c = \frac{R_f}{\rho g} Q_M + \frac{1}{\rho g} P_U - \frac{R_f}{\rho g} Q_U \\
\tau_p = \frac{A_T R_f}{\rho g} \quad \text{process time constant}
\]
Float Level Sensor

- Assume a zero-order dynamic model, i.e., the dynamics are negligible relative to the process time constant $\tau_p$ since the cross-sectional area of the tank is assumed large.
- Consider the actual dynamics to justify this assumption:

$$F_B = W_f + A_f \rho g (h - x)$$

steady condition: $h = x$
- **Equation of Motion**

\[
F_B - W_f - B \frac{dx}{dt} = M_f \frac{d^2 x}{dt^2}
\]

\[
M_f \frac{d^2 x}{dt^2} + B \frac{dx}{dt} = A_f \rho g (h - x)
\]

\[
M_f \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + A_f \rho g x = A_f \rho g h
\]

\[
\left[ \frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1 \right] x = Kh
\]

\[
\omega_n = \sqrt{\frac{A_f \rho g}{M_f}}
\]

\[
\zeta = \frac{B}{2 \sqrt{A_f M_f \rho g}}
\]

\[
K = 1.0
\]
– To measure rapid changes in \( h \) accurately, \( \omega_n \) must be sufficiently large. The specific weight of the fluid \( (\rho g) \) is not a design variable, so strive for large values of \( A_f/M_f \) (i.e., hollow floats).

– In our case, the tank has a large diameter and if the inflow and outflow rates are modest, \( h \) cannot change rapidly and so a zero-order model is justified.

• **Potentiometer Bridge and Electronic Amplifier**
  – Obviously these two components are fast enough to be treated as zero order in this system.
This device produces a pneumatic output signal closely proportional (± 5% nonlinearity) to an electrical input (± 5V and 3-15 psig).

We are concerned with overall dynamics from $e_A$ to $p$. The block diagram shows a 4th-order open-loop differential equation.

However, experimental frequency response tests show typically a flat amplitude ratio out to about 5 Hz. This response is very fast relative to $\tau_p$ so we model the electropneumatic transducer as zero order.
• **Pneumatic Valve Positioner**
  – We are only interested in the overall dynamics relating $x_v$ to $p$. These are again quite fast relative to $\tau_p$, so we model the component as zero order.
  – The valve positioner allows one to “characterize” the static calibration curve between $p$ and $x_v$ and thus obtain desired linear or nonlinear relationships between $p$ and manipulated flowrate $Q_M$.

• **Relation between $Q_M$ and $x_v$**
  – This relationship is assumed to be statically linear and dynamically instantaneous and thus a zero-order model.
  – Although the dynamic response of $Q_M$ to $x_v$ is not instantaneous due to fluid inertia and compliance, the response is much faster than the tank-filling dynamics.
• Closed-Loop System Block Diagram and Differential Equation

\[
\begin{align*}
\left[ (K_h h_V - K_h h_C) K_a K_p K_x K_v + \frac{1}{R_f} P_U - Q_U \right] \frac{R_f}{\rho g} \frac{1}{\tau_p s + 1} = h_C \\
(\tau_s s + 1) h_C &= \frac{K}{K + 1} h_V + \frac{1}{\rho g (K + 1)} P_U - \frac{R_f}{\rho g (K + 1)} Q_U \\
\tau_s &= \frac{\tau_p}{K + 1} \quad \text{Closed-Loop System Time Constant} \\
K &= \frac{1}{\rho g} \left( K_h K_a K_p K_x K_v R_f \right) \quad \text{System Loop Gain}
\end{align*}
\]
• **Speed of Response**
  
  – Response for a step input in $h_v$ (hold perturbations $P_U$ and $Q_U$ at zero)

  $$h_C = \frac{K}{K+1} h_{v_s} \left( 1 - e^{-\frac{t}{\tau_s}} \right)$$

  – Response for a step input in disturbances $P_U$ and $Q_U$ (hold $h_v = 0$)

  $$h_C = \frac{1}{\rho g (K+1)} P_{U_s} \left( 1 - e^{-\frac{t}{\tau_s}} \right)$$

  $$h_C = \frac{-R_f}{\rho g (K+1)} Q_{U_s} \left( 1 - e^{-\frac{t}{\tau_s}} \right)$$

  – Increasing loop gain $K$ increases the speed of response

  $$\tau_s = \frac{\tau_p}{K+1} = \text{closed-loop system time constant}$$
• **Steady-State Errors**
  - A procedure generally useful for all types of systems and inputs is to rewrite the closed-loop system differential equation with system error \( (V-C) \), rather than the controlled variable \( C \), as the unknown.
  - In this case we have:

\[
(\tau_s s + 1) h_C = \frac{K}{K + 1} h_v + \frac{1}{\rho g(K + 1)} P_U - \frac{R_f}{\rho g(K + 1)} Q_U
\]

\[
h_E = h_V - h_C
\]

\[
(\tau_s s + 1)(h_V - h_E) = \frac{K}{K + 1} h_v + \frac{1}{\rho g(K + 1)} P_U - \frac{R_f}{\rho g(K + 1)} Q_U
\]

\[
(\tau_s s + 1) h_E = \left(\tau_s s + \frac{1}{K + 1}\right) h_v - \frac{1}{\rho g(K + 1)} P_U + \frac{R_f}{\rho g(K + 1)} Q_U
\]
For any chosen commands or disturbances, the steady-state error will just be the particular solution of the equation:

\[
(\tau_s s + 1) h_E = \left(\tau_s s + \frac{1}{K + 1}\right) h_V - \frac{1}{\rho g (K + 1)} P_U + \frac{R_f}{\rho g (K + 1)} Q_U
\]

We see that the steady-state error is improved if we increase the loop gain K.

For any initial equilibrium condition we can “trim” the system for zero error but subsequent steady commands and/or disturbances must cause steady-state errors.

Ramp inputs would cause steady-state errors that increase linearly with time, the rate of increase being proportional to ramp slope and inversely proportional to K+1.
• **Stability**
  – All aspects of system behavior are improved by increasing loop gain – up to a point! – instability may result, but our present model gives no warning of this. Why?
  – We neglected dynamics in some components and a general rule is:
    If we want to make valid stability predictions we must include enough dynamics in our system so that the closed-loop system differential equation is at least third order. The one exception is systems with dead times where instability can occur even when dynamics are zero, first, or second order.
  – Is our model then useless?
No! It does correctly predict system behavior as long as the loop gain is not made “too large.” As $K$ is increased, the closed-loop system response gets faster and faster. At some point, the neglected dynamics are no longer negligible and the model becomes inaccurate.

We neglected dynamics relative to $\tau_p$, but in the closed-loop system, response speed is determined by $\tau_s$.

**Exercise:**

- Compare the responses of the following 3 systems for $K = 1, 5, 10$ with $\tau_1 = 1.0$, $\tau_2 = 0.1$, and $\tau_3 = 0.05$. The input is a unit step.
- Examine speed of response, steady-state error, and stability predictions.
MatLab / Simulink Diagram
Liquid Level Control

Response for $K = 1$

- C1
- C2
- C3

response: C1 C2 C3

time (sec)
Response for $K = 5$

![Graph showing response for K = 5 with points C1, C2, C3 marked on the plot.](image)
Response for $K = 10$

- **C1**
- **C2**
- **C3**
Response of C1 for $K = 1, 5, 10$

- $K=1$
- $K=5$
- $K=10$
Response of C2 for $K = 1, 5, 10$

- $K=1$
- $K=5$
- $K=10$
Response of C3 for K = 1, 5, 10

K=1

K=5

K=10

time (sec)
Note that as loop gain $K$ is increased, the speed of response is increased and the steady-state error is reduced.

For what value of loop gain $K$ will any of these systems go unstable?

Let’s look at the closed-loop system transfer functions and characteristic equations:

<table>
<thead>
<tr>
<th>Transfer Functions</th>
<th>Characteristic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1 \frac{K}{V} = \frac{K}{\tau_1 s + 1 + K}$</td>
<td>$(\text{The characteristic equation is obtained by setting the denominator polynomial equal to zero.})$</td>
</tr>
<tr>
<td>$C_2 \frac{K}{V} = \frac{K}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2) s + 1 + K}$</td>
<td></td>
</tr>
<tr>
<td>$C_3 \frac{K(\tau_3 + 1)}{V} = \frac{K(\tau_3 + 1)}{\tau_1 \tau_2 \tau_3 s^3 + (\tau_1 \tau_2 + \tau_2 \tau_3 + \tau_1 \tau_3) s^2 + (\tau_1 + \tau_2 + \tau_3) s + 1 + K}$</td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{align*}
\frac{C_1}{V} &= \frac{K}{s + 1 + K} \\
\frac{C_2}{V} &= \frac{K}{0.1s^2 + 1.1s + 1 + K} \\
\frac{C_3}{V} &= \frac{K(0.05s + 1)}{0.005s^3 + 0.155s^2 + 1.15s + 1 + K}
\end{align*}
\]

- The only system which will go unstable as the loop gain $K$ is increased is the third system; its characteristic equation is third order. The first two systems will continue to show improved speed of response and reduction of steady-state error without any hint of instability!
Let’s apply the three methods of determining closed-loop system stability – Routh, Nyquist, and Root-Locus - to the third system and determine the value of $K$ for which this system becomes marginally stable.

**Routh Stability Criterion**

- **Closed-Loop System Characteristic Equation**
  
  \[0.005s^3 + 0.155s^2 + 1.15s + 1 + K = 0\]

- **Routh Array**

  \[
  \begin{array}{cc}
  0.005 & 1.15 \\
  0.155 & 1 + K \\
  (0.155)(1.15) - (1 + K)(0.005) & 0 \\
  0.155 & 0 \\
  1 + K & 0 \\
  \end{array}
  \]
• For stability we see that:
  \[(0.155)(1.15) - (1 + K)(0.005) > 0\]
  \[1 + K > 0\]
• This leads to the result that for absolute stability:
  \[-1 < K < 34.65\]
• A simulation with the loop gain set to \(K = 34.65\) should verify this result. The value of gain \(K = -1\) will give the closed-loop system characteristic equation a root at the origin but that value is of less interest, since we rarely use negative gain values.
• Note that at the loop gain value of 34.65, only system 3 is marginally stable. Systems 1 and 2 show no signs of instability, only improved speed of response and reduced steady-state error.
Response for $K = 34.65$

- C1
- C2
- C3

Time (sec)

Response: C1, C2, C3
Root-Locus Interpretation of Stability

- The open-loop transfer function is:

\[
\frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}
\]

\[
= \frac{K}{\tau_1 \tau_2 \tau_3 s^3 + (\tau_1 \tau_2 + \tau_2 \tau_3 + \tau_1 \tau_3) s^2 + (\tau_1 + \tau_2 + \tau_3) s + 1}
\]

\[
= \frac{K}{0.005s^3 + 0.155s^2 + 1.15s + 1}
\]

- The root-locus plot shows that when \(K = 34.65\), the system is marginally stable. For that value of \(K\), the closed-loop poles are at: \(-31\) and \(\pm 15.1658i\).
Root-Locus Plot for $K > 0$

$K = 34.65$

Open-Loop Poles: 
-20, -10, -1
– Nyquist Stability Criterion

• A polar plot of the open-loop transfer function for the gain $K = 34.65$ goes through the point $-1$, indicating marginal stability of the closed-loop system.

• A polar plot of the open-loop transfer function for the gain $K = 10$ shows a gain margin $= 3.46$.

• The Bode plots for a gain $K = 10$ show a gain margin $= 10.794 \text{ dB} = 3.46$ and a phase margin $= 40.5$ degrees.
Nyquist Diagrams
From: U(1)
To: Y(1)

\[ K = 34.65 \]

- \( \omega = 0^- \)
- \( \omega = 0^+ \)
- \( \omega = +\infty \)
- \( \omega = -\infty \)
Nyquist Diagrams

From: $U(1)$

To: $Y(1)$

$K = 10$

$-0.289$

Gain Margin = $1 / 0.289 = 3.46$
$K = 10$

Bode Diagrams

$G_m = 10.794 \, \text{dB} \, \text{at} \, 15.166 \, \text{rad/sec}, \ P_m = 40.539 \, \text{deg. at} \, 7.4486 \, \text{rad/sec}$

$GM = 10.8 \, \text{dB}
PM = 40.5 \, \text{deg}$
• **Saturation**
  – The amplifier, electropneumatic transducer, and valve positioner all exhibit saturation, limiting their output when the input becomes too large.
  – All real systems must exhibit such power limitations and one of the consequences is that the closed-loop response speed improvement will not be realized for large signals.

• **Exercise:**
  – Simulate the following two systems for gains of \( K = 1, 10 \) and a unit step input.
  – Examine speed of response and steady-state error.
MatLab / Simulink Diagram
C1 and C2 responses are identical.
Unit Step Response for C1 and C2 with $K = 10$

The speed of response is affected, but the steady-state error is not.
Step Response of Magnitude 1.5 for C1 and C2 with $K = 10$

Both speed of response and steady-state error are affected.
Let’s make the system model more realistic by modeling the pneumatic valve positioner as a first-order system:

\[
\frac{x_v}{p}(D) = \frac{K_x}{\tau_{vp}D + 1}
\]

The open-loop system is now second order. The closed-loop system differential equation is now:

\[
\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} + 1 \right)h_c = \frac{K}{K + 1}h_v + \frac{\tau_{vp}D + 1}{\rho g (K + 1)}P_U - \frac{R_f (\tau_{vp}D + 1)}{\rho g (K + 1)}Q_U
\]

\[
\omega_n = \sqrt{\frac{K + 1}{\tau_p \tau_{vp}}}
\]

\[
\zeta = \frac{\tau_p + \tau_{vp}}{2 \sqrt{\tau_p \tau_{vp} (K + 1)}}
\]

\[
K = \frac{1}{\rho g} \left( K_h K_a K_p K_x K_v R_f \right)
\]
To get fast response (large $\omega_n$) for given lags $\tau_p$ and $\tau_{vp}$, we must increase loop gain $K$. How does $K$ affect $\zeta$?

If $\tau_p = 60$ sec and $\tau_{vp} = 1.0$ sec and we desire $\zeta = 0.6$, what is $K$? What is $\omega_n$? Is absolute instability possible with this model? What does the Nyquist plot show as $K$ is increased? What does the root-locus plot show as $K$ is increased?

• Consider Gain Distribution
  – How does gain distribution affect stability and dynamic response of the closed-loop system?
  – Are steady-state errors for disturbances sensitive to gain distribution?
  – Should one optimize the distribution of gain so as to minimize steady-state errors?
\[ G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ s_{1,2} = -\zeta\omega_n \pm i\omega_n \sqrt{1 - \zeta^2} \]

\[ s_{1,2} = -\sigma \pm i\omega_d \]

\[ y(t) = 1 - e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \]

\[ t_r \approx \frac{1.8}{\omega_n} \text{ rise time} \]

\[ t_s \approx \frac{4.6}{\zeta\omega_n} \text{ settling time} \]

\[ M_p = e^{\sqrt{1 - \zeta^2}} \quad (0 \leq \zeta < 1) \text{ overshoot} \]

\[ \approx \left( 1 - \frac{\zeta}{0.6} \right) \quad (0 \leq \zeta \leq 0.6) \]
\[ \omega_n \geq \frac{1.8}{t_r} \]

\[ \zeta \geq 0.6(1 - M_p) \quad 0 \leq \zeta \leq 0.6 \]

\[ \sigma \geq \frac{4.6}{t_s} \]

**Time-Response Specifications vs. Pole-Location Specifications**
**Integral Control**

- Consider integral control of this liquid-level process. Replace the amplifier block $K_a$ by $K_I/s$.
- The closed-loop system differential equation is:

$$
\left[ (h_V - h_C) \frac{K_h K_L K_p K_x K_v}{s} + \frac{1}{R_f} P_U - Q_U \right] \frac{R_f}{\rho g} \frac{\rho g}{\tau_p s + 1} = h_C
$$

$$
\left( \tau_p D^2 + D + K \right) h_C = K h_V + \frac{1}{\rho g} D P_U - \frac{R_f}{\rho g} D Q_U
$$

$$
\left( \tau_p D^2 + D + K \right) h_E = -\left( \tau_p D^2 + D \right) h_V + \frac{1}{\rho g} D P_U - \frac{R_f}{\rho g} D Q_U
$$

$$
K = \frac{K_h K_L K_p K_x K_v R_f}{\rho g} \text{ loop gain} \quad h_E = h_V - h_C \text{ system error}
$$
– Note the following:
– Step changes (constant values) of $h_V$, $P_U$, and/or $Q_U$ give zero steady-state errors.
– For ramp inputs, we now have constant, nonzero steady-state errors whose magnitudes can be reduced by increasing $K$.
– The characteristic equation is second order, so define:

$$\omega_n = \sqrt{\frac{K}{\tau_p}} \quad \zeta = \frac{1}{2\sqrt{K\tau_p}}$$

– If we take $\tau_p$ as unavailable for change, we see that an increase in $K$ to gain response speed or decrease ramp steady-state errors will be limited by loss of relative stability (low $\zeta$).
– If we design for a desired $\zeta$, the needed $K$ is easily found and $\omega_n$ is then fixed.
– Absolute instability is not predicted; the model is too simple.
– See the comparison between proportional control and integral control: root locus plots and Nyquist plots. Note the destabilizing effects of integral control.
Proportional Control

\[ \frac{K}{\tau_p s + 1} \]

Integral Control

\[ \frac{K}{s(\tau_p s + 1)} \]

Root Loci

Nyquist Plots
• What if we change the controlled process by closing off the pipe on the left side of the tank. This not only deletes $P_U$ as a disturbance but also causes a significant change in process dynamics.

• Take $R_f = \infty$. The from Conservation of Mass we have:

$$h_c = \frac{1}{A_T} (Q_M - Q_U)$$

• The original tank process had self-regulation.
  – If one changes $Q_M$ and/or $Q_U$ the tank will itself in time find a new equilibrium level since the flow through $R_f$ varies with level.
– With $R_f$ not present, the tiniest difference between $Q_M$ and $Q_U$ will cause the tank to completely drain or overflow since it is now an integrator and has lost its self-regulation.

– Even with proportional control, the integrating effect in the process gives zero steady-state error for step commands, but not for disturbances.

– If we substitute integral control to eliminate the $Q_U$ error, the system becomes absolutely unstable.