Modeling, Analysis, & Control of Dynamic Systems

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21st-Century Multidisciplinary Systems Engineering Education

Modern Mechatronic System

Other Components
Communications

Actuation
Power Modulation
Energy Conversion

Computation
Software, Electronics

Operator Interface
Human Factors

Instrumentation
Energy Conversion
Signal Processing

Physical System
Mechanical, Fluid,
Thermal, Chemical,
Electrical, Mixed
Educational Challenge

- Control Design and Implementation is still the domain of the specialist.
- Controls and Electronics are still viewed as afterthought add-ons.
- Very few practicing engineers perform any kind of physical and mathematical modeling.
- Mathematics is a subject that is not viewed as enhancing one’s engineering skills but as an obstacle to avoid.
- Very few engineers have the balance between analysis and hardware essential for success in Mechatronics.
Balance: The Key to Success

Modeling & Analysis

Experimental Validation & Hardware Implementation

The Mechatronic Design Process

Computer Simulation Without Experimental Verification Is At Best Questionable, And At Worst Useless!
Do you ever feel like this?

Engineering Problem

Systematic, Structured Approach to Design

Integrated Design Concept

Sensors
Electronics
Controls

Model, Analyze & Predict

Computers
Actuators
Mechanical

Build & Test

YES!

NO!

Cost-Effective, High-Quality, Timely, Robust Design
The cornerstone of modern engineering practice!

**Engineering System Investigation Process**

- **Physical System**
  - System Measurement
  - Measurement Analysis
  - Comparison: Predicted vs. Measured
  - Design Changes

- **Parameter Identification**
  - Physical Model
  - Mathematical Model
  - Mathematical Analysis

**Flowchart Diagram:**

1. Start with Physical System
2. Measure System
3. Analyze Measurement
4. Compare Predicted vs. Measured
5. Design Changes
6. Check Adequacy of Comparison
   - If Yes, proceed further
   - If No, adjust model and repeat
Physical & Mathematical Modeling

Less Real, Less Complex, More Easily Solved

Truth Model Design Model

More Real, More Complex, Less Easily Solved

Hierarchy Of Models
Always Ask: Why Am I Modeling?
Electro-Dynamic Vibration Exciter
Physical System vs. Physical Model

Physical System

Physical Model

Modeling, Analysis, & Control
of Dynamic Systems

K. Craig
• This "moving coil" type of device converts an electrical command signal into a mechanical force and/or motion and is very common, e.g., vibration shakers, loudspeakers, linear motors for positioning heads on computer disk memories, and optical mirror scanners.

• In all these cases, a current-carrying coil is located in a steady magnetic field provided by permanent magnets in small devices and electrically-excited wound coils in large ones.

• Two electromechanical effects are observed in such configurations:
  – **Generator Effect**: Motion of the coil through the magnetic field causes a voltage proportional to velocity to be induced into the coil.
  – **Motor Effect**: Passage of current through the coil causes it to experience a magnetic force proportional to the current.
• Flexure $K_f$ is an intentional soft spring (stiff, however, in the radial direction) that serves to guide the axial motion of the coil and table.

• Flexure damping $B_f$ is usually intentional, fairly strong, and obtained by laminated construction of the flexure spring, using layers of metal, elastomer, plastic, and so on.

• The coupling of the coil to the shaker table would ideally be rigid so that magnetic force is transmitted undistorted to the mechanical load. Thus $K_t$ (generally large) and $B_t$ (quite small) represent parasitic effects rather than intentional spring and damper elements.

• $R$ and $L$ are the total circuit resistance and inductance, including contributions from both the shaker coil and the amplifier output circuit.
Equations of Motion

\[ M_t \ddot{x}_t = -K_f x_t - B_f \dot{x}_t + B_t \left( \ddot{x}_c - \dot{x}_t \right) + K_t \left( x_c - x_t \right) \]

\[ M_c \ddot{x}_c = -B_t \left( x_c - x_t \right) - K_t \left( x_c - x_t \right) + Ki \]

\[ L \dot{i} = e_i - Ri - K \dot{x}_c \]

State-Space Representation

\[
\begin{bmatrix}
\dot{x}_t \\
\ddots \\
\dot{x}_t \\
\vdots \\
\dot{x}_c \\
\ddots \\
\dot{x}_c \\
\vdots \\
i
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-K_f & -K_t & -B_f & -B_t & K_t & B_t \\
M_t & M_t & M_t & M_t & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
K_t & B_t & -K_t & -B_t & K & 0 \\
M_c & M_c & M_c & M_c & M_c & M_c \\
0 & 0 & 0 & -K & -R & 0 \\
0 & 0 & 0 & 0 & 1 & L \\
i
\end{bmatrix}
\begin{bmatrix}
x_t \\
\vdots \\
x_t \\
\vdots \\
x_c \\
\vdots \\
x_c \\
\vdots \\
i
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
e_i \\
0 \\
0 \\
1 \\
L
\end{bmatrix}
\]
Typical Parameter Values (SI Units)
L = 0.0012
R = 3.0
K = 190
K_t = 8.16E8
B_t = 3850
K_f = 6.3E5
B_f = 1120
M_c = 1.815
M_t = 6.12
5 Elements of an Observer

Sensor output $Y(s)$
Plant Excitation $P_C(s)$
Plant Model $G_{PEst}(s)$
Sensor Model $G_{SEst}(s)$
Observer Compensator $G_{CO}(s)$

One application of the observer is to use the observed state to close the control loop.
Electro-Pneumatic Transducer

Block Diagram of an Electro-Pneumatic Transducer

Input Voltage $e_{in}$

Coil Current $i_c$

Magnetic Force $f_m$

Flapper Motion $x_f$

Output Pressure $p_o$

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This system can be collapsed into a simplified approximate overall model when numerical values are properly chosen:

\[
\frac{p_o}{e_{in}} = \frac{K}{\tau D + 1}
\]

**Differential Operator** \( D = \frac{d}{dt} \)
• It is interesting to note here that while the block diagram shows one input for the system, i.e., command voltage $e_{in}$, there are possible undesired inputs that must also be considered.

• For example, the ambient temperature will affect the electric coil resistance, the permanent magnet strength, the leaf-spring stiffness, the damper-oil viscosity, the air density, and the dimensions of the mechanical parts. All these changes will affect the system output pressure $p_o$ in some way, and the cumulative effects may not be negligible.
Temperature Feedback Control System: A Larger-Scale Engineering System
• Note that in the block diagram of this system, the detailed operation of the electropneumatic transducer is not made apparent; only its overall input/output relation is included.
• The designer of the temperature feedback-control dynamic system would consider the electropneumatic transducer an off-the-shelf component with certain desirable operating characteristics.
• The methods of system dynamics are used by both the electropneumatic transducer designer and the designer of the larger temperature feedback-control system.
Diagram Showing How Physical Model Hardware Parameters Are Related to Physical Model Dynamic System Performance
Cantilever Beam
Mechanical System

Steel Cantilever Beam
Eddy-Current Damper
Strain Gage

Accelerometer

Vibration Exciter

Hard-Drive Read-Write Head

MEMS Accelerometer

Modeling, Analysis, & Control of Dynamic Systems
Stepper Motor System Design: Ink-Jet Printer Application

Stepper Motor Open-Loop and Closed-Loop Control

Experimental System

Engineering Application
Mechatronics Module: Smart Actuator
NI ELVIS

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ELVIS Connections
- Circuit Input to FUNC OUT
- Measured Signal to Oscilloscope CH B+
- Oscilloscope CH B- to Power Ground
- Circuit Ground to Power Ground

RC Circuit
Step Response
**ELVIS Connections**

- FUNC OUT to Circuit Input
- Measured Signal to Analog Input Signal ACH0+
- FUNC OUT to Analog Input Signal ACH1+
- Power Ground to Analog Input Signals ACH0- and ACH1- and Circuit Ground

**RC Circuit Frequency Response**

[Graph showing frequency response of an RC circuit]
**RC Electrical System**

\[ e_{in} - e_R - e_C = 0 \]

\[ e_{in} - iR - e_{out} = 0 \]

\[ e_{in} - \left( C \frac{de_{out}}{dt} \right) R - e_{out} = 0 \]

\[ RC \frac{de_{out}}{dt} + e_{out} = e_{in} \]

\[ \frac{e_{out}}{e_{in}} = \frac{1}{RCD + 1} \quad \tau = RC \]

**Spring-Damper Mechanical System**

\[ f_i - f_B - f_K = 0 \]

\[ f_i - Bv - Kx = 0 \]

\[ f_i - Bv - f_o = 0 \]

\[ f_i - B \left( \frac{f_o}{K} \right) - f_o = 0 \]

\[ \frac{B}{K} \dot{f_o} + f_o = f_i \]

\[ f_o = \frac{1}{B} \frac{D + 1}{K} \quad \tau = \frac{B}{K} \]
Balancing Robot
Nonlinear Equations of Motion:

\[
\left( \overline{I} + m\overline{r}^2 \right)\ddot{\theta} - B \frac{\dot{x}}{r} + m\overline{r} \cos \theta - mg\overline{r} \sin \theta = -T_d + T_f
\]

\[
\left( m + m_w + \frac{\overline{I}_w}{r^2} \right) \ddot{x} \quad (m\overline{r} \cos \theta) \ddot{\theta} + m\overline{r} \dot{\theta}^2 \sin \theta \quad \left( \frac{B}{r^2} \right) \dot{x} = \frac{1}{r} (T_d - T_f)
\]

Simplifying Assumptions:

- Two degrees of freedom: x and \( \theta \)
- Wheels roll without slipping
- Rotating structure is a rigid body
- Both wheel motors mounted to rotating body are identical
- Rate gyro and inclinometer give instantaneous response
Magnetic Levitation System

Electromagnetic Valve Actuator For a Camless Automotive Engine

Modeling, Analysis, & Control of Dynamic Systems
The diagram depicts a magnetic levitation system, which is a genuine mechatronic system. The system consists of an electromagnet, an infrared LED, a phototransistor, and a levitated ball. The equations for forces in the system are given by:

\[ f(x, i) = C \left( \frac{i}{x} \right)^2 \]

where \( f \) represents the force, \( i \) is the current, \( x \) is the distance, and \( C \) is a constant.
Rotary Inverted Pendulum
Controlled System Response

Rotary Inverted Pendulum

SG Gains
- Theta Gain: 0.21
- Theta Dot Gain: 0.299
- Alpha (Pendulum) Gain: 4.4514
- Alpha Dot Gain: 0.0203

System Gain
- Swing-Up Gain

Control Select
- State Space
- Classical

Motor Power

Output to Amp
- -0.63582

Control State
- Balancing
- Deadzone
- Swingup
- Swingup
NI Week 2006
Keynote Presentation

Modeling, Analysis, & Control of Dynamic Systems

K. Craig
Graphical System Design Applied

A unique design team at the Institute for Systems Research at the University of Maryland, led by Dr. Craig Craig, has developed a new tool for modeling and analyzing dynamic systems. The team's goal was to design a vehicle that could be used for human transportation, but also for remote-controlled exploration. The solution was a two-wheeled, balancing transport platform that could be controlled remotely using LabVIEW.

To test the system, the team used the vehicle to perform experiments in the laboratory and on campus. Using the software, the team was able to simulate the behavior of the system and make improvements. The system was then tested in a real-world environment, and the team was able to show that the system was able to perform as expected.

The design team attributes its success to the use of graphical system design and LabVIEW. The team was able to complete the project in four months, and the final system was able to meet all of the requirements set forth by the design criteria.

— Dr. Andrew Craig, Designer

Sensor Fusion

The team also addressed the challenge of sensor fusion, which is the ability to combine data from multiple sensors to improve system performance. The team used a variety of sensors, including accelerometers, gyroscopes, and magnetometers, to create a robust system that could accurately determine the position and orientation of the vehicle.

The team worked closely with industry partners to ensure that the system was able to meet the needs of its intended applications. The team was able to demonstrate the effectiveness of the system in a variety of test scenarios, including urban and rural environments.

The team's approach to sensor fusion was innovative and unique, and it has the potential to revolutionize the field of dynamic system design. The team's success in developing the system is a testament to their hard work and dedication.

— Dr. Andrew Craig, Designer
Integration and Assessment Early in the Design Process

Fast Component Mounter Placement Module

Modeling, Analysis, & Control of Dynamic Systems
Engineering System Investigation Process

The cornerstone of modern engineering practice!
Engineering System Investigation Overview

- Apply the steps in the process when:
  - An actual physical system exists and one desires to understand and predict its behavior.
  - The physical system is a concept in the design process that needs to be analyzed and evaluated.
- After recognizing a need for a new product or service, one generates design concepts by using:
  - Past experience (personal and vicarious)
  - Awareness of existing hardware
  - Understanding of physical laws
  - Creativity
• The importance of modeling and analysis in the design process has never been more important.
• Design concepts can no longer be evaluated by the build-and-test approach because it is too costly and time consuming.
• Validating the predicted dynamic behavior in this case, when no actual physical system exists, then becomes even more dependent on one's past hardware and experimental experience.
Do you ever feel like this?

Engineering Problem

Systematic, Structured Approach to Design

Integrated Design Concept

Electronics
Sensors
Controls

Model, Analyze & Predict

Build & Test

Computers
Actuators
Mechanical

YES!

Cost-Effective, High-Quality, Timely, Robust Design

NO!

Questions?
• **Dynamic Physical System**
  – Any collection of interacting elements for which there are cause-and-effect relationships among the time-dependent variables. The present output of the system depends on past inputs.

• **Analysis of the Dynamic Behavior of Physical Systems**
  – Cornerstone of modern technology
  – More than any other field links the engineering disciplines

• **Purpose of a Dynamic System Investigation**
  – Understand & predict the dynamic behavior of a system
  – Modify and/or control the system, if necessary
Essential Features of the Study of Dynamic Systems

- Deals with entire operating machines and processes rather than just isolated components.
- Treats dynamic behavior of mechanical, electrical, electromechanical, fluid, thermal, chemical, and mixed systems.
- Emphasizes the behavioral similarity between systems that differ physically and develops general analysis and design tools useful for all kinds of physical systems.
- Sacrifices detail in component descriptions so as to enable understanding of the behavior of complex systems made from many components.
– Uses methods which accommodate component descriptions in terms of experimental measurements, when accurate theory is lacking or is not cost-effective, and develops universal lab test methods for characterizing component behavior.

– Serves as a unifying foundation for many practical application areas, e.g., vibrations, measurement systems, control systems, acoustics, vehicle dynamics, etc.

– Offers a wide variety of computer software to implement its methods of analysis and design.
Balance: The Key to Success

Modeling & Analysis

Experimental Validation & Hardware Implementation

The Mechatronic Design Process

Computer Simulation Without Experimental Verification Is At Best Questionable, And At Worst Useless!
Balance in Engineering is the Key!

- The essential characteristic of an engineer and the key to success is a *balance* between the following sets of skills:
  - modeling (physical and mathematical), analysis (closed-form and numerical simulation), and control design (analog and digital) of dynamic physical systems
  - experimental validation of models and analysis and understanding the key issues in hardware implementation of designs
• **Stages of a Dynamic System Investigation**
  
  – **Physical System**
    • Define the physical system to be studied, along with the system boundaries, input variables, and output variables.
  
  – **Physical System to Physical Model**
    • In general, a physical model is an imaginary physical system which resembles an actual system in its salient features, but which is simpler, more ideal, and is thereby more amenable to analytical studies. There is a hierarchy of physical models of varying complexity possible.
    • Not oversimplified, not overly complicated - *a slice of reality.*
• The astuteness with which approximations are made at the onset of an investigation is the very crux of engineering analysis.

• The ability to make shrewd and viable approximations which greatly simplify the system and still lead to a rapid, reasonably accurate prediction of its behavior is the hallmark of every successful engineer.

• What is the purpose of the model? Develop a set of performance specifications for the model based on the specific purpose of the model. What features must be included? How accurately do they need to be represented?
The challenges to physical modeling are formidable:

- Dynamic behavior of many physical processes is complex.
- Cause and effect relationships are not easily discernible.
- Many important variables are not readily identified.
- Interactions among the variables are hard to capture.

Engineering Judgment is the Key!
– In modeling dynamic systems, we consider matter and energy as being continuously, though not necessarily uniformly, distributed over the space within the system boundaries.

– This is the macroscopic or continuum point of view. We consider the system variables as quantities which change continuously from point to point in the system as well as with time and this always leads to a distributed-parameter physical model which results in a partial differential equation mathematical model.

– Models in this most general form behave most like the real systems at the macroscopic level.
Because of the mathematical complexity of these models, engineers find it necessary and desirable to work with less exact models in many cases.

Simpler models which concentrate matter and energy into discrete lumps are called lumped-parameter physical models and lead to ordinary differential equation mathematical models.

An understanding of the difference between distributed-parameter and lumped-parameter models is vital to the intelligent formulation and use of lumped models.

The time-variation of the system parameters can be random or deterministic, and if deterministic, variable or constant.
Comments on Truth Model vs. Design Model

– In modeling dynamic systems, we use engineering judgment and simplifying assumptions to develop a physical model. The complexity of the physical model depends on the particular need, and the intelligent use of simple physical models requires that we have some understanding of what we are missing when we choose the simpler model over the more complex model.

– The truth model is the model that includes all the known relevant characteristics of the real system. This model is often too complicated for use in engineering design, but is most useful in verifying design changes or control designs prior to hardware implementation.
The design model captures the important features of the process for which a control system is to be designed or design iterations are to be performed, but omits the details which you believe are not significant.

In practice, you may need a hierarchy of models of varying complexity:

- A very detailed truth model for final performance evaluation before hardware implementation
- Several less complex truth models for use in evaluating particular effects
- One or more design models
Physical Modeling

Less Real, Less Complex, More Easily Solved

Truth Model  Design Model

More Real, More Complex, Less Easily Solved

Hierarchy Of Models
Always Ask: Why Am I Modeling?
<table>
<thead>
<tr>
<th><strong>Approximation</strong></th>
<th><strong>Mathematical Simplification</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Neglect small effects</td>
<td>Reduces the number and complexity of the equations of motion</td>
</tr>
<tr>
<td>Assume the environment is independent of system motions</td>
<td>Reduces the number and complexity of the equations of motion</td>
</tr>
<tr>
<td>Replace distributed characteristics with appropriate lumped elements</td>
<td>Leads to ordinary (rather than partial) differential equations</td>
</tr>
<tr>
<td>Assume linear relationships</td>
<td>Makes equations linear; allows superposition of solutions</td>
</tr>
<tr>
<td>Assume constant parameters</td>
<td>Leads to constant coefficients in the differential equations</td>
</tr>
<tr>
<td>Neglect uncertainty and noise</td>
<td>Avoids statistical treatment</td>
</tr>
</tbody>
</table>
• **Neglect Small Effects**
  
  – Small effects are neglected on a *relative* basis. In analyzing the motion of an airplane, we are unlikely to consider the effects of solar pressure, the earth's magnetic field, or gravity gradient. To ignore these effects in a space vehicle problem would lead to grossly incorrect results!

• **Independent Environment**
  
  – In analyzing the vibration of an instrument panel in a vehicle, we assume that the vehicle motion is *independent* of the motion of the instrument panel.
• **Lumped Characteristics**
  – In a lumped-parameter model, system dependent variables are assumed uniform over finite regions of space rather than over infinitesimal elements, as in a distributed-parameter model. Time is the only independent variable and the mathematical model is an ordinary differential equation.
  – In a distributed-parameter model, time and spatial variables are independent variables and the mathematical model is a partial differential equation.
  – Note that elements in a lumped-parameter model do not necessarily correspond to separate physical parts of the actual system.
A long electrical transmission line has resistance, inductance, and capacitance distributed continuously along its length. These distributed properties are approximated by lumped elements at discrete points along the line.

It is important to note that a lumped-parameter model, which may be valid in low-frequency operations, may not be valid at higher frequencies, since the neglected property of distributed parameters may become important. For example, the mass of a spring may be neglected in low-frequency operations, but it becomes an important property of the system at high frequencies.
• **Linear Relationships**
  – Nearly all physical elements or systems are inherently nonlinear if there are no restrictions at all placed on the allowable values of the inputs, e.g., saturation, dead-zone, square-law nonlinearities.
  – If the values of the inputs are confined to a sufficiently small range, the original nonlinear model of the system may often be replaced by a linear model whose response closely approximates that of the nonlinear model.
  – When a linear equation has been solved once, the solution is general, holding for all magnitudes of motion.
– Linear systems also satisfy conditions for superposition.

– The superposition property states that for a system initially at rest with zero energy: (1) multiplying the inputs by any constant multiplies the outputs by the same constant, and (2) the response to several inputs applied simultaneously is the sum of the individual responses to each input applied separately.

• **Constant Parameters**
  – Time-varying systems are ones whose characteristics change with time.
  – Physical problems are simplified by the adoption of a model in which all the physical parameters are constant.
• **Neglect Uncertainty and Noise**
  - In real systems we are uncertain, in varying degrees, about values of parameters, about measurements, and about expected inputs and disturbances. Disturbances contain random inputs, called noise, which can influence system behavior.
  - It is common to neglect such uncertainties and noise and proceed as if all quantities have definite values that are known precisely.
Classes of Systems

- Distributed Parameter
- Lumped Parameter

- Stochastic
- Deterministic

- Discrete Time
- Continuous Time

- Nonlinear
- Linear

- Time Varying
- Constant Coefficient

- Non-Homogeneous
- Homogeneous
• The most realistic physical model of a dynamic system leads to differential equations of motion that are:
  – nonlinear, partial differential equations with time-varying and space-varying parameters
  – These equations are the most difficult to solve.
• The simplifying assumptions discussed lead to a physical model of a dynamic system that is less realistic and to equations of motion that are:
  – linear, ordinary differential equations with constant coefficients
  – These equations are easier to solve and design with.
Physical Model to Mathematical Model

- We derive a mathematical model to represent the physical model, i.e., write down the differential equations of motion of the physical model.
- The goal is a generalized treatment of dynamic systems, including mechanical, electrical, electromechanical, fluid, thermal, chemical, and mixed systems.
  - Define System: Boundary, Inputs, Outputs
  - Define Variables: Through and Across Variables
  - Write System Relations: Dynamic Equilibrium Relations and Compatibility Relations
  - Write Constitutive Relations: Physical Relations for Each Element
  - Combine: Generate State Equations
- **Study Dynamic Behavior**
  - Study the dynamic behavior of the mathematical model by solving the differential equations of motion either through mathematical analysis or computer simulation.
  - Dynamic behavior is a consequence of the system structure - don’t blame the input!
  - Seek a relationship between physical model structure and behavior.
  - Develop **insight** into system behavior.
- **Comparison: Actual vs. Predicted**
  - Compare the predicted dynamic behavior to the measured dynamic behavior from tests on the actual physical system; make physical model corrections, if necessary.

- **Make Design Decisions**
  - Make design decisions so that the system will behave as desired:
    - modify the system (e.g., change the physical parameters of the system, add a sensor, change the actuator or its location)
    - control the system (e.g., augment the system, typically by adding a dynamic system called a compensator or controller)
Dynamic System Investigation Example: Spring-Mass System

Mechanical oscillations are very important physical phenomena.

A spring-mass system is used in the modeling and analysis of many engineering systems. It is often embedded in real systems, or a real system may sometimes be modeled as a spring-mass system due to its similar dynamic behavior.
• **Physical System**
  
  – A mass hanging at the tip of a tension spring that is attached to a stationary rigid support constitutes the dynamic system for investigation.
  
  – The motion of the spring-mass system is constrained by a linear bearing on the side of the support so that the mass oscillates only in one direction, the vertical direction.
  
  – A non-contact optical (infrared) sensor is used to measure the position of the mass.
  
  – The free oscillation of the spring-mass system is considered here; there is no externally-applied driving force acting on the system.
• **Physical-Model Simplifying Assumptions**
  - The support to which the spring is attached is rigid. This assumption in effect says that the environment is independent of system motions.
  - The spring is pure, i.e., it only has the characteristic (elasticity) for which it is named. A pure spring has negligible mass and damping. This, of course, is an idealization as all springs have mass and dissipate energy upon cycling. If the spring mass is less than 10% of the mass attached to it, this is a reasonable assumption (except in high-speed applications). The energy dissipation in the spring is very small compared to other dissipation mechanisms in the system, so neglecting it is also reasonable.
– The spring is ideal, i.e., there is a linear relationship between spring force and spring displacement in the range of mass motion considered. This can be experimentally verified. The actual spring is a tension spring with some pretension (a force that pulls the coils of the spring together) and the motion of the mass must be restricted to the range during which the spring is in tension, i.e., large amplitudes of motion of the mass are excluded from consideration.

– The attached mass can be treated as a rigid body.

– The mass moves with one degree of freedom in pure translation in the vertical plane. There is no out-of-plane motion and there is no rotational motion of the mass. The mass then can be treated as a point mass.
The friction in the system is parasitic, i.e., no energy dissipation mechanism has been designed into the system. Any air damping due to the motion of the mass in the air is negligible. The friction in the linear bearing is the main source of energy dissipation and, based on engineering experience, is a combination of viscous damping (proportional to the velocity of the mass and directed opposite to the mass motion) and dry-friction, or Coulomb damping, as it is called, (essentially constant in magnitude, independent of mass velocity, and directed opposite to the mass motion). Coulomb friction leads to a nonlinear mathematical model, while viscous friction leads to a linear mathematical model.
The desire to have a linear mathematical model does not justify the assumption of viscous damping and the omission of Coulomb damping. If this assumption is not based on sound engineering judgment, then the resulting mathematical model will not predict the actual behavior of the dynamic system. However, initially this parasitic damping will be neglected so as to keep the analysis simpler. If it is determined that this assumption is invalid, then a viscous damping model will be assumed with an experimentally-determined viscous damping coefficient.
- The system is vertical with the acceleration due to gravity pointing downward and constant in value.
- All parameters (mass $m$, spring constant $k$, viscous damping coefficient $B$) are constant, i.e., do not change with time or temperature, for example.
Spring

Graph of Force vs. Displacement
Pure and Ideal Spring
(2 data points are used for plot)

- $F_{spring} (N)$
- $2^{nd}$ Weight (N)
- Actual Weight = $mg (N)$
- Y-Axis Intercept = Spring Pre-Tension $F_t (N)$
- Static Equilibrium Displacement (m)

Slope of Straight Line = Spring Constant $K (N/m)$

$F = K (x_2 - x_1)$
Physical Model

- Physical Model

\[ m = \text{mass} \]
\[ m_s = \text{spring mass} \]
\[ L = \text{unstretched spring length} \]
\[ k = \text{spring constant} \]
\[ g = \text{acceleration due to gravity} \]
\[ x_s = \text{static spring stretch} \]
\[ x_d = \text{dynamic spring stretch} \]
\[ x = \text{total spring stretch} = x_s + x_d \]
• **Parameter Identification**
  – Parameters in the physical model that need to be identified are:
    • Mass $m$ (kg) of the attached block - obtained by weighing the block
    • Mass $m_s$ (kg) of the spring - obtained by weighing the spring
    • Unstretched length of the spring $L$ (m) - obtained by direct measurement
    • Spring constant $k$ (N/m) of the spring
    • Pretension force $F_t$ (N) of the spring
Two different masses must be used to obtain two data points (weight and spring displacement) so that the assumed linear (straight-line) behavior of the spring can be plotted.

Experimental Determination of Spring Constant $k$ and Pretension Force $F_t$
Parameter Identification Test Results

- Unstretched spring length $L = 0.127$ m
- Spring constant $k = 491.1$ N/m
- Spring pretension force $F_t = 9.521$ N
- Mass $m = 5.231$ kg
- Spring mass $m_s = 0.08$ kg
- Static equilibrium displacement $= 0.0851$ m
- Equation of force $F_{spring}$ (N) vs. displacement $x$ (m) curve is: $F_{spring} = 491.1x + 9.5206$ and this linear curve is valid in the range 0 to 0.120 meters spring displacement.
• **Mathematical Model**
  – Newton’s 2\textsuperscript{nd} Law of Motion
  
  – Free-Body Diagrams

\[
\sum F_x = m \frac{d^2 x}{dt^2} = m \ddot{x}
\]

Static Equilibrium

Motion
- **Equations of Motion**

\[
mg - (kx_s + F_t) = 0
\]

\[
x_s = \frac{mg - F_t}{k} \quad \text{Static Equilibrium Position}
\]

\[
mg - \left[ k(x_s + x_d) + F_t \right] = m\ddot{x}
\]

\[
m\ddot{x} + k(x_s + x_d) = mg - F_t
\]

\[
m\ddot{x} + kx = mg - F_t \quad \text{Equation of Motion in terms of } x = x_s + x_d
\]

\[
m\ddot{x}_d + kx_d = 0 \quad \text{Equation of Motion in terms of } x_d
\]
Predicted Dynamic Response

Initial Mass Displacement is 60 mm from the Static Equilibrium Position

\[ \omega_n = \sqrt{\frac{k}{m}} = 9.69 \text{ rad/sec} \]

\[ f_n = \frac{\omega_n}{2\pi} = 1.54 \text{ cycles/sec or Hz} \]
Actual Dynamic Response

Initial Mass Displacement is 60 mm from the Static Equilibrium Position.

System damping has caused the oscillations to decay to zero in about 25 seconds.
• **Comparison: Actual vs. Predicted**
  
  – The frequency of the oscillations for both responses is approximately the same, i.e., 1.5 Hz. This is as expected from engineering experience since parasitic damping has little effect on the natural frequency of oscillations of the system.

  – The oscillations of the mass in both the predicted response and the actual response is about the static equilibrium position, i.e., $x_s = 85.1$ mm.

  – The amplitude of the oscillations for both responses does not agree well at all. This also is as expected from engineering experience because even a small amount of damping will reduce the amplitude of oscillations significantly over time. The greater the damping, the faster the oscillations decay to zero.
Viscous Fluid Damper

\[ F = B(v_2 - v_1) \]
Viscous Damper
(Piston/Cylinder)

A relative velocity between the cylinder and piston forces the viscous oil through the clearance space $h$, shearing the fluid and creating a damping force.

$$\mu = \text{fluid viscosity}$$

$$B = \frac{6\pi\mu L}{h^3} \left[ \left( R_2 - \frac{h}{2} \right)^2 - R_1^2 \right] \left[ \frac{R_2^2 - R_1^2}{R_2 - \frac{h}{2}} - h \right]$$
Coulomb Friction Model

"Stiction" Coulomb Friction Model
Example of Coulomb Friction

\[ m = 0.1 \text{ kg} \]
\[ k = 100 \text{ N/m} \]
\[ F_{\text{stick}} = 0.25 \text{ N} \]
\[ F_{\text{slip}} = 0.20 \text{ N} \]
\[ V_0 = 0.002 \text{ m/sec} = \text{constant} \]
• **Physical Model Modification**
  
  - Viscous damping is now included in the physical model.
  - Revised Free-Body Diagrams

\[
\begin{align*}
\text{(a)} & \quad \text{Static Equilibrium} \\
\text{(b)} & \quad \text{Motion}
\end{align*}
\]
Revised Equations of Motion

\[ mg - B\ddot{x} - \left[ k(x_s + x_d) + F_t \right] = m\ddot{x} \]

\[ m\dddot{x} + B\ddot{x} + k(x_s + x_d) = mg - F_t \]

\[ m\dddot{x} + B\ddot{x} + kx = mg - F_t \quad \text{Equation of Motion in terms of } x = x_s + x_d \]

\[ m\dddot{x}_d + B\ddot{x}_d + kx_d = 0 \quad \text{Equation of Motion in terms of } x_d \]
Physical Model of the Spring-Mass System
Spring-Mass System
Comparison: Predicted Dynamic Response vs. Actual Dynamic Response

Initial Mass Displacement is 60 mm from the Static Equilibrium Position

experimentally-determined viscous-damping coefficient:
B = 1.1 N/(m/s)
- The agreement between the predicted response and the actual response is now quite good.
- The difference between the two responses becomes more noticeable as the oscillations diminish.
- This is expected because, again from engineering experience, Coulomb friction will begin to dominate the response over viscous friction when the system begins to slow down, and the physical model does not contain a Coulomb-friction term.
• **Conclusions**
  
  – The dynamic system investigation process has been demonstrated for the simple spring-mass system. Even for such a simple mechanical system, the process is still quite involved.
  
  – Many simplifying assumptions must be made and sound engineering judgment used. The result of the investigation is a physical and mathematical model that together predict quite well the behavior of the actual physical system.
  
  – If a more detailed model is needed to more accurately predict the behavior of the actual system, the simplifying assumptions need to be reexamined and modified, if necessary, e.g., include Coulomb friction from the linear bearing, account for the mass of the spring.
Spring-Pendulum Physical System
Spring-Pendulum Dynamic System Investigation
Physical Model
Simplifying Assumptions

- pure spring, i.e., negligible inertia and damping
- ideal (linear) spring
- frictionless pivot
- neglect all material damping and air damping
- point mass, i.e., neglect rotational inertia of mass
- two degrees of freedom, i.e., $r$ and $\theta$ are the generalized coordinates (this assumes no out-of-plane motion and no bending of the spring)
- support structure is rigid
Physical Model with Parameter Identification

\( m = \text{pendulum mass} = 1.815 \text{ kg} \)

\( m_{spring} = \text{spring mass} = 0.1445 \text{ kg} \)

\( \ell = \text{unstretched spring length} = 0.333 \text{ m} \)

\( k = \text{spring constant} = 172.8 \text{ N/m} \)

\( g = \text{acceleration due to gravity} = 9.81 \text{ m/s}^2 \)

\( F_t = 5.71 \text{ N} = \text{pre-tension of spring} \)

\( r_s = \text{static spring stretch, i.e., } r_s = (mg-F_t)/k = 0.070 \text{ m} \)

\( r_d = \text{dynamic spring stretch} \)

\( r = \text{total spring stretch} = r_s + r_d \)
Spring Parameter Identification

\[ F_{spring} (N) \]

mg = 17.805 N

Spring Pre-Tension
\[ F_t = 5.71 \text{ N} \]

0.070 m

K = 172.8 N/m

Spring Displacement
(meters)
Polar Coordinates: Position, Velocity, Acceleration

\[ \ddot{r} = r \ddot{e}_r \]

\[ \dot{\mathbf{v}} = \frac{\mathbf{d} \dot{r}}{\mathbf{d}t} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta = v_r \hat{e}_r + v_\theta \hat{e}_\theta \]

\[ \ddot{\mathbf{a}} = \frac{\mathbf{d} \dot{\mathbf{v}}}{\mathbf{d}t} = \left( \ddot{r} - r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} + 2r \dot{\theta} \right) \hat{e}_\theta \]

\[ = a_r \hat{e}_r + a_\theta \hat{e}_\theta \]

\[ \dddot{r} \quad \text{magnitude change} \]
\[ \dddot{\theta} \quad \text{direction change} \]
\[ r \dddot{\theta} + r \dot{\theta} \quad \text{magnitude change} \]
\[ r \ddot{\theta}^2 \quad \text{direction change} \]
Rigid Body Kinematics

XY: R reference frame (ground)
xy: $R_1$ reference frame (pendulum)

\[
\begin{bmatrix}
x \\
y \\
z \\
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix}
\]

\[
\vec{R} \vec{a}^P = \vec{R} \vec{a}^O + \left[ \vec{R} \vec{\omega}^{R_1} \times \left( \vec{R} \vec{\omega}^{R_1} \times \vec{r}^{OP} \right) \right] + \left[ \vec{R} \vec{\alpha}^{R_1} \times \vec{r}^{OP} \right] + \vec{R}_1 \vec{a}^P + 2 \left[ \vec{R} \vec{\omega}^{R_1} \times \vec{r}^{OP} \right]
\]
Rigid Body Kinematics

\[ \mathbf{R} \mathbf{a}^O = 0 \]
\[ \mathbf{R} \mathbf{\dot{\omega}}^{R_1} = \dot{\theta} \mathbf{k} = \dot{\theta} \mathbf{K} \]
\[ \mathbf{r}^{OP} = - (\ell + r) \mathbf{j} = - (\ell + r) \left[ -\sin \theta \mathbf{I} + \cos \theta \mathbf{J} \right] \]
\[ \mathbf{R} \mathbf{\ddot{\alpha}}^{R_1} = \ddot{\theta} \mathbf{k} = \ddot{\theta} \mathbf{K} \]
\[ \mathbf{R}_1 \mathbf{v}^P = - \mathbf{\dot{r}} \mathbf{j} = - \mathbf{r} \left[ -\sin \theta \mathbf{I} + \cos \theta \mathbf{J} \right] \]
\[ \mathbf{R}_1 \mathbf{a}^P = - \mathbf{\dot{r}} \mathbf{j} = - \mathbf{r} \left[ -\sin \theta \mathbf{I} + \cos \theta \mathbf{J} \right] \]

After substitution and evaluation:

\[ \mathbf{R} \mathbf{a}^P = \mathbf{i} \left[ (\ell + r) \ddot{\theta} + 2 \mathbf{r} \dot{\theta} \right] + \mathbf{j} \left[ -\mathbf{r} + (\ell + r) \dot{\theta}^2 \right] \]
Mathematical Model

\[ \sum F_r = ma_r = m\left[ \ddot{r} - (\ell + r)\dot{\theta}^2 \right] \]
\[ \sum F_\theta = ma_\theta = m\left[ (\ell + r)\ddot{\theta} + 2\dot{r}\dot{\theta} \right] \]

\[
\begin{align*}
    m\ddot{r} - m(\ell + r)\dot{\theta}^2 + kr + F_t - mg \cos \theta &= 0 \\
    (\ell + r)\ddot{\theta} + 2\dot{r}\dot{\theta} + g \sin \theta &= 0
\end{align*}
\]

\[
\begin{align*}
    -kr - F_t + mg \cos \theta &= m\left[ \ddot{r} - (\ell + r)\dot{\theta}^2 \right] \\
    -mg \sin \theta &= m\left[ (\ell + r)\ddot{\theta} + 2\dot{r}\dot{\theta} \right]
\end{align*}
\]

Free Body Diagram

Nonlinear Equations of Motion
Mathematical Model: Lagrange’s Equations

\[
\begin{align*}
q_1 &= r \\
q_2 &= \theta \\
T &= \frac{1}{2} m \left[ i^2 + (\ell + r)^2 \dot{\theta}^2 \right] \\
V &= \frac{1}{2} kr^2 - mg \left[ (\ell + r) \cos \theta - \ell \right] \\
m \ddot{r} - m (\ell + r) \dot{\theta}^2 + kr + F_t - mg \cos \theta &= 0 \\
(\ell + r) \ddot{\theta} + 2 \dot{r} \dot{\theta} + g \sin \theta &= 0
\end{align*}
\]

Lagrange’s Equations

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i
\]

Generalized Coordinates

Generalized Forces

\[
\begin{align*}
Q_r &= -F_t \\
Q_\theta &= 0
\end{align*}
\]
Simulation Results

**Initial Conditions**

\[ \theta_0 = -0.274 \text{ rad} \]
\[ r_0 = 0.046 \text{ m} \]
Simulation Results

Initial Conditions

$\theta_0 = 0.021 \text{ rad}$

$r_0 = 0.115 \text{ m}$
Actual Measured Dynamic Behavior

Initial Conditions

\[ \theta_0 = -0.274 \text{ rad} \]
\[ r_0 = 0.046 \text{ m} \]
Actual Measured Dynamic Behavior

**Initial Conditions**

\[ \theta_0 = 0.021 \text{ rad} \]

\[ r_0 = 0.115 \text{ m} \]

Experimental Results with Initial Conditions: \( \theta = 0.021 \text{ rad}, r = 0.115 \text{ m} \)
Comparison

Simulation Results with Initial Conditions: $\theta = -0.274 \text{ rad}, r = 0.046 \text{ m}$

Experimental Results with Initial Conditions: $\theta = -0.274 \text{ rad}, r = 0.046 \text{ m}$

Initial Conditions:

$\theta_0 = -0.274 \text{ rad}$
$r_0 = 0.046 \text{ m}$
Comparison

Simulation Results with Initial Conditions: \( \theta_0 = 0.021 \text{ rad}, r_0 = 0.115 \text{ m} \)

Experimental Results with Initial Conditions: \( \theta_0 = 0.021 \text{ rad}, r_0 = 0.115 \text{ m} \)

Initial Conditions:
\[ \theta_0 = 0.021 \text{ rad} \]
\[ r_0 = 0.115 \text{ m} \]
LabVIEW Control Design

Magnetic Levitation System

Electromagnet
Infrared LED
Phototransistor
Levitated Ball

Magnetic Levitation Control Design with LabVIEW

\[ f(x, i) = C \left( \frac{i^2}{x^3} \right) \]

Electromagnet

Ball (mass m)

\[ mg \]

\[ i \]

\[ \Sigma \]

\[ V_{\text{desired}} \]

\[ V_{\text{bias}} \]

\[ G_c(s) \text{ Controller} \]

\[ \Sigma \]

\[ \Sigma \]

\[ \Sigma \]

\[ i \]

\[ G(s) \text{ Magnet + Ball} \]

\[ X \]

\[ V_{\text{actual}} \]

\[ H(s) \text{ Sensor} \]

---

Modeling, Analysis, & Control of Dynamic Systems

K. Craig 113
Magnetic Levitation System

Electromagnetic Valve Actuator
For a Camless Automotive Engine
Magnetic Levitation System - A Genuine Mechatronic System

Electromagnet

Infrared LED

Levitated Ball
- m = 0.008 kg
- r = 0.0062 m = 0.24 in

Phototransistor
- $V_{\text{sensor}} = 5.44 \, \text{V}$
- At Equilibrium

Equilibrium Conditions
- $x_0 = 0.003 \, \text{m}$
- $i_0 = 0.222 \, \text{A}$
• Electromagnet Actuator
  – Current flowing through the coil windings of the electromagnet generates a magnetic field.
  – The ferromagnetic core of the electromagnet provides a low-reluctance path in which the magnetic field is concentrated.
  – The magnetic field induces an attractive force on the ferromagnetic ball.

**Electromagnetic Force**
Proportional to the square of the current and inversely proportional to the square of the gap distance

\[ f(x, i) = C \left( \frac{i^2}{x^2} \right) \]
- The electromagnet uses a $\frac{1}{4}$-inch steel bolt as the core with approximately 3000 turns of 26-gauge magnet wire wound around it.
- The resistance of the electromagnet at room temperature is approximately 32 $\Omega$. 
\[ \phi = \phi_{l} + \phi_{m} \quad \text{Neglect} \quad \phi_{l} \quad \phi_{m} = \frac{N\text{i}}{R_{m}} \]

\[ \lambda = N\phi = N\phi_{m} = \frac{N^2\text{i}}{R_{m}} = L_{m}\text{i} \quad R_{m} = R_{\text{core}} + R_{\text{gap}} + R_{\text{object}} + R_{\text{return path}} \]

Define: \( R = R_{\text{core}} + R_{\text{object}} + R_{\text{return path}} = \text{constant} \)

\[ R_{\text{gap}} = \frac{x_{\text{gap}}}{\mu_0 A_{\text{gap}}} \quad L_{m} = \frac{N^2}{R_{m}} = \frac{N^2}{R + \frac{x_{\text{gap}}}{\mu_0 A_{\text{gap}}}} = \frac{\mu_0 A_{\text{gap}} N^2}{\mu_0 A_{\text{gap}} R + x_{\text{gap}}} \]

\[ W_{\text{field}} = \frac{1}{2} L(x) i^2 = \frac{1}{2} \frac{\mu_0 A_{\text{gap}} N^2}{\mu_0 A_{\text{gap}} R + x_{\text{gap}}} i^2 \]

\[ f_{c} = \frac{1}{2} i^2 \frac{dL(x)}{dx} = -\frac{1}{2} \mu_0 A_{\text{gap}} N^2 \left( \frac{1}{\mu_0 A_{\text{gap}} R + x_{\text{gap}}} \right)^2 = -K_{1} \left( \frac{i}{K_{2} + x_{\text{gap}}} \right)^2 \]

"Modeling, Analysis, & Control of Dynamic Systems"  
K. Craig 118
Ball-Position Sensor

LED Blocked: $V_{sensor} = 0 \text{ V}$
LED Unblocked: $V_{sensor} = 10 \text{ V}$
Equilibrium Position: $V_{sensor} \approx 5.40 \text{ V}$
$K_{sensor} \approx 4 \text{ V/mm}$ \quad Range $\pm 1\text{mm}$

$i_{emitter} = 15 \text{ mA}$

Modeling, Analysis, & Control of Dynamic Systems
K. Craig 119
• **Ball-Position Sensor**
  – The sensor consists of an infrared diode (emitter) and a phototransistor (detector) which are placed facing each other across the gap where the ball is levitated.
  – Infrared light is emitted from the diode and sensed at the base of the phototransistor which then allows a proportional amount of current to flow from the transistor collector to the transistor emitter.
  – When the path between the emitter and detector is completely blocked, no current flows.
  – When no object is placed between the emitter and detector, a maximum amount of current flows.
  – The current flowing through the transistor is converted to a voltage potential across a resistor.
– The voltage across the resistor, $V_{\text{sensor}}$, is sent through a unity-gain, follower op-amp to buffer the signal and avoid any circuit loading effects.

– $V_{\text{sensor}}$ is proportional to the vertical position of the ball with respect to its operating point; this is compared to the voltage corresponding to the desired ball position.

– The emitter potentiometer allows for changes in the current flowing through the infrared LED which affects the light intensity, beam width, and sensor gain.

– The transistor potentiometer adjusts the phototransistor current-to-voltage conversion sensitivity and allows adjustment of the sensor’s voltage range; a 0 - 10 volt range allows for maximum sensor sensitivity without saturation of the downstream buffer op-amp.
Magnetic Levitation System Block Diagram

Linear Feedback Control System to Levitate Steel Ball about an Equilibrium Position Corresponding to Equilibrium Gap $x_0$ and Equilibrium Current $i_0$

From Equilibrium:
As $i \uparrow$, $x \downarrow$, & $V_{\text{sensor}} \downarrow$
As $i \downarrow$, $x \uparrow$, & $V_{\text{sensor}} \uparrow$

Electromagnet

$$ f(x, i) = C \left( \frac{i^2}{x^2} \right) $$

Ball (mass m)

$mg$
Assume Ideal Op-Amp Behavior

\[ e^+ = e^- \]

Voltage-to-Current Converter

\[ i_M = \left( \frac{R_2}{R_1 + R_2} \right) \left( \frac{1}{R_S} \right) e_{in} \]

\[ R_1 = 49K\Omega, \quad R_2 = 1K\Omega, \quad R_3 = 0.1\Omega \]
Non-Ideal Op-Amp Behavior

\[ e_o = \frac{A}{\tau s + 1} \left( e^+ - e^- \right) \]

\[ e_{out} - e_1 = \left( L_M s + R_M \right) i \]

\[ e_1 = R_S i \]

\[ e_{out} - e_1 = \left( L_M s + R_M \right) \frac{e_1}{R_S} \]

\[ e_{out} = \left( \frac{L_M s + R_M + R_S}{R_S} \right) e_1 \]
**Magnetic Levitation System**

**Control System Design**

Equation of Motion:

\[
\ddot{m}x = mg - C\left(\frac{i^2}{x^2}\right)
\]

At Equilibrium:

\[
mg = C\left(\frac{i^2}{x^2}\right)
\]

Linearization:

\[
C\left(\frac{i^2}{x^2}\right) \approx C\left(\frac{i^2}{\bar{x}^2}\right) - C\left(\frac{2i^2}{\bar{x}^3}\right)\dot{x} + C\left(\frac{2i}{\bar{x}^2}\right)\dot{i}
\]

\[
\ddot{m}x = mg - C\left(\frac{i^2}{x^2}\right) + C\left(\frac{2i^2}{\bar{x}^3}\right)\dot{x} - C\left(\frac{2i}{\bar{x}^2}\right)\dot{i}
\]

\[
\ddot{m}x = C\left(\frac{2i^2}{\bar{x}^3}\right)\dot{x} - C\left(\frac{2i}{\bar{x}^2}\right)\dot{i}
\]
Use of Experimental Testing in Multivariable Linearization

\[ f_m = f(i, y) \]

\[ f_m \approx f(i_0, y_0) + \left. \frac{\partial f}{\partial y} \right|_{i_0, y_0} (y - y_0) + \left. \frac{\partial f}{\partial i} \right|_{i_0, y_0} (i - i_0) \]
\[
\begin{align*}
\dot{x} &= 6540\ddot{x} - 88\dot{i} \\
mg &= C \left( \frac{i^2}{x^2} \right) \\
C &= 1.4332 \times 10^{-5} \\
m &= 0.008 \\
g &= 9.81 \\
x &= 0.003 \\
\bar{i} &= 0.222 \\
K_{\text{sensor}} &= 4 \text{ V/mm} \\
K_{\text{amp}} &= 0.2 \text{ A/V}
\end{align*}
\]
Open-Loop Transfer Function

\[ \frac{88}{(s^2 - 6540)(0.2)(4000)} = \frac{70400}{s^2 - 6540} \]

Controller

PD Controller

\[ \frac{K_p + K_d s}{\tau s + 1} \]

\[ \frac{K}{s + \frac{K_p}{K_d}} \]

\[ \frac{1}{\tau s + \frac{1}{\tau}} \]

\[ \tau = 0.002 \]

\[ K_p = 0.3 \]

\[ K_d = 0.003 \]

Open-Loop Bode Plot

Root Locus Plot
Active Lead Controller

\[
\begin{align*}
-V_{\text{error}} &= \frac{1}{R_1} \left[ \frac{R_1 C_1 s + 1}{R_2 C_2 s + 1} \right] - \frac{R_4}{R_3} \\
V_{\text{control}} &= \left[ \frac{R_4}{R_3} \right] \left[ \frac{0.01s + 1}{0.001s + 1} \right]
\end{align*}
\]

Modeling, Analysis, & Control of Dynamic Systems
K. Craig 131
Control System Introduction

- Open-Loop Control
  - Basic Open-Loop Control
  - Input-Compensated Feedforward Control
- Closed-Loop (Feedback) Control
- Process Control Example: Feedback and/or Feedforward
- Closed-Loop Systems Can Go Unstable! Why?
- Feedback Control System Design Procedure
- Series vs. Parallel Compensation
- Analog vs. Digital Control
Introduction to Control Systems

• Process or Plant
• Process Inputs
  • Manipulated Inputs
  • Disturbance Inputs
• Response Variables

Control systems are an integral part of the overall system and not after-thought add-ons! The earlier the issues of control are introduced into the design process, the better!

Why Controls?
• Command Following
• Disturbance Rejection

Everything Needs Controls for Optimum Functioning!
• Classification of Control System Types
  – Open-Loop
    • Basic
    • Input-Compensated Feedforward
      – Disturbance-Compensated
      – Command-Compensated
  – Closed-Loop (Feedback)
    • Classical
      – Root-Locus
      – Frequency Response
    • Modern (State-Space)
    • Advanced
      – e.g., Adaptive, Nonlinear, Fuzzy Logic
**Basic Open-Loop Control System**

Satisfactory if:
- disturbances are not too great
- changes in the desire value are not too severe
- performance specifications are not too stringent
Open-Loop Input-Compensated Feedforward Control: Disturbance-Compensated

- Measure the disturbance
- Estimate the effect of the disturbance on the controlled variable and compensate for it

Modeling, Analysis, & Control of Dynamic Systems

K. Craig 136
Open-Loop Input-Compensated Feedforward Control: Command-Compensated

Based on the knowledge of plant characteristics, the desired value input is augmented by the command compensator to produce improved performance.
• Open-loop systems without disturbance or command compensation are generally the simplest, cheapest, and most reliable control schemes. These should be considered first for any control task.
• If specifications cannot be met, disturbance and/or command compensation should be considered next.
• When conscientious implementation of open-loop techniques by a knowledgeable designer fails to yield a workable solution, the more powerful feedback methods should be considered.
Closed-Loop (Feedback) Control System

Open-Loop Control System is converted to a Closed-Loop Control System by adding:

- measurement of the controlled variable
- comparison of the measured and desired values of the controlled variable
Generalized Operational Block Diagram of a Feedback Control System
Open-Loop Control System

V → \( K_A \) → \( K_C \) → \( \sum \) → \( \frac{K_p}{\tau_p D + 1} \) → C

- Desired Value
- Reference Input
- Controller
- Manipulated Variable
- Disturbance Input Element
- Disturbance
- Controlled Variable

Closed-Loop Control System

V → \( K_A \) → \( \sum \) → \( K_C \) → \( \sum \) → \( \frac{K_p}{\tau_p D + 1} \) → C

- Feedback Signal
- Actuating Signal
- Error
- Sensor Error
- Sensor (Feedback Element)
• **Basic Benefits of Feedback Control**
  
  – Cause the controlled variable to accurately follow the desired variable; corrective action occurs as soon as the controlled variable deviates from the command.
  
  – Greatly reduces the effect on the controlled variable of all external disturbances in the forward path. It is ineffective in reducing the effect of disturbances in the feedback path (e.g., those associated with the sensor), and disturbances outside the loop (e.g., those associated with the reference input element).
  
  – Is tolerant of variations (due to wear, aging, environmental effects, etc.) in hardware parameters of components in the forward path, but not those in the feedback path (e.g., sensor) or outside the loop (e.g., reference input element).
– Can give a closed-loop response speed much greater than that of the components from which they are constructed.

• **Inherent Disadvantages of Feedback Control**
  – No corrective action is taken until after a deviation in the controlled variable occurs. Thus, perfect control, where the controlled variable does not deviate from the set point during load or set-point changes, is theoretically impossible.
  – It does not provide predictive control action to compensate for the effects of known or measurable disturbances.
– It may not be satisfactory for processes with large time constants and/or long time delays. If large and frequent disturbances occur, the process may operate continually in a transient state and never attain the desired steady state.

– In some applications, the controlled variable cannot be measured on line and, consequently, feedback control is not feasible.

• For situations in which feedback control by itself is not satisfactory, significant improvements in control can be achieved by adding feedforward control.
• **Disturbance-Compensated Feedforward Control**
  
  – **Basic Idea**: Measure important load variables and take corrective action *before* they upset the process.
  
  – In contrast, a feedback controller does not take corrective action until after the disturbance has upset the process and generated an error signal.
  
  – There are several disadvantages to disturbance-compensated feedforward control:
    
    • The load disturbances must be measured on line. In many applications, this is not feasible.
The quality of the feedforward control depends on the accuracy of the process model; one needs to know how the controlled variable responds to changes in both the load and manipulated variables.

Ideal feedforward controllers that are theoretically capable of achieving perfect control may not be physically realizable. Fortunately, practical approximations of these ideal controllers often provide very effective control.
**Process Control Example**

**Objective:**
Ensure that $T$ remains at or near the set point.

Inlet temperature $T_i$ changes with time.

Continuous Stirred-Tank Heater

$dm_i/dt$  
$T_i$  
$dm/dt$  
$T$
• **Assumptions:**
  – Inlet and outlet flow rates are identical and liquid density is constant; therefore, volume $V$ of liquid in the tank is constant
  – Tank is perfectly mixed and heat losses are negligible; therefore, the exit temperature is equal to the temperature of the liquid in the tank
  – Specific Heat, $C$, of the fluid is constant
  – Steady-State Energy Balance:

$$
\bar{Q} = \left( \frac{dm}{dt} \right) \bar{C} (\bar{T} - \bar{T}_i)
$$

• The bar indicates the nominal steady-state design values
• Suppose the inlet temperature $T_i$ changes with time. How can we ensure that $T$ remains at or near the set point $T_R$?

• **Possible strategies** for controlling exit temperature $T$:
  - Measure $T$, adjust $Q$: Feedback Control
  - Measure $T_i$, adjust $Q$: Feedforward Control
  - Measure $T$, adjust $dm/dt$: Feedback Control
  - Measure $T_i$, adjust $dm/dt$: Feedforward Control
  - Measure $T$ and $T_i$, adjust $Q$: Feedback and Feedforward Control
  - Measure $T$ and $T_i$, adjust $dm/dt$: Feedback and Feedforward Control
• What about other sources of process disturbances?

• **Feedforward Control**
  – In principle, feedforward control is capable of providing perfect control in the sense that the controlled variable is maintained at the set point.
  – Corrective action taken cancels out the effects of the disturbances before the controlled variable is affected.
  – However, no corrective action is taken for unmeasured disturbances.

• **Feedback Control**
  – Feedback control is not capable of perfect control since the controlled variable must deviate from the set point before corrective action is taken.
– However, corrective action is taken regardless of the source of the disturbance.
– The ability to handle unmeasured disturbances of unknown origin is a major reason why feedback controllers are so widely used.
Boiler Drum Level Feedback Control System

The level of the boiling liquid is measured and used to adjust the feedwater flow rate.
• **Boiler Drum Level Control System: Feedback Control**

  – The level of the boiling liquid is measured and used to adjust the feedwater flow rate.

  – This control system tends to be quite sensitive to rapid changes in the load variable (the steam flow rate) caused by steam demands made by downstream processing units, due to the small liquid capacity of the boiler drum.

  – Large controller gains cannot be used since level measurements exhibit rapid fluctuations for boiling liquids and a high controller gain would tend to amplify the measurement noise and produce unacceptable variations in the feedwater flow rate.
Steam flow rate is measured and the feedforward controller adjusts the feedwater flow rate so as to balance the steam demand.
• Boiler Drum Level Control System: Feedforward Control
  – The feedforward control scheme can provide better control of the liquid level.
  – The steam flow rate is measured and the feedforward controller adjusts the feedwater flow rate so as to balance the steam demand.
  – The controlled variable, liquid level, is not measured.
  – As an alternative, steam pressure could be measured instead of steam flow rate.
Boiler Drum Level Feedforward & Feedback Control System
**Boiler Drum Level Control System: Feedforward and Feedback Control**

- In practical applications, feedforward control is normally used in combination with feedback control.
- Feedforward control is used to reduce the effects of measurable disturbances.
- Feedback control compensates for inaccuracies in the process model, measurement errors, and unmeasured disturbances.
- The diagram shows one possible combination of feedback and feedforward control.
Instability in Feedback Control Systems

• All feedback systems can become unstable if improperly designed.
• In all real-world components there is some kind of lagging behavior between the input and output, characterized by $\tau$’s and $\omega_n$’s.
• Instantaneous response is impossible in the real world!
• Instability in a feedback control system results from an improper balance between the strength of the corrective action and the system dynamic lags.
Consider the following example:
Liquid level $C$ in a tank is manipulated by controlling the volume inflow rate $M$ by means of a 3-position on/off controller. Transfer function $K/D$ between $M$ and $C$ represents conservation of volume between inflow rate and liquid level. Liquid-level sensor measures $C$ perfectly but with a data transmission delay, $\tau_{DT}$.

Tank Liquid-Level Feedback Control System
Stable Behavior of the Tank Liquid-Level Feedback Control System

- Signal C: solid
- Signal B: dotted
- Signal 0.1*M: dashed
Unstable Behavior of the Tank Liquid-Level Feedback Control System

- Signal C: solid
- Signal B: dotted
- Signal 0.1*M: dashed

Time (sec)
Feedback Control System Design Procedure

- Control Engineering is an important part of the design process of most dynamic systems.
- The deliberate use of feedback can:
  - Stabilize an otherwise unstable system
  - Reduce the error due to disturbance inputs
  - Reduce the tracking error while following a command input
  - Reduce the sensitivity of a closed-loop transfer function to small variations in internal system parameters
• Remember that the purpose of control is to aid the product or process – the mechanism, the robot, the chemical plant, the aircraft, or whatever – to do its job.

• **Engineers must take into account early in their plans the contribution of control to the design process!** More and more systems are being designed so that they will not work without feedback!

• Control system design begins with a proposed product or process whose satisfactory dynamic performance depends on feedback for:
  - Stability
  - Disturbance Regulation
  - Tracking Accuracy
  - Reduction of the Effects of Parameter Variations
Having a general step-by-step approach for feedback control system design serves two purposes:

- It provides a useful starting point for any real-world controls problem.
- It provides meaningful checkpoints once the design process is underway.
• **Sequence of Steps for Feedback Control System Design**

1. **Understand the process and translate dynamic performance requirements into time, frequency, or pole-zero specifications.**
   - The importance of understanding the process cannot be overemphasized!
   - Do not confuse the approximation with the reality!
   - You must be able to:
     - Use the simplified model for its intended purpose
     - Return to an accurate model or the actual physical system to really verify the design performance
2. **Select the types and number of sensors considering location, technology, functional performance, physical properties, quality factors, and cost.**

- Which variables are important to control?
- Which variables can physically be measured?
- Select sensors that indirectly allow a good estimate to be made of the critical unmeasurable variables.
- It is important to consider sensors for the disturbances, e.g., in chemical processes, it is often beneficial to sense a load disturbance directly because improved performance can be obtained if this information is fed forward to the controller.
3. Select the types and number of actuators considering location, technology, functional performance, physical properties, quality factors, and cost.

- In order to control a dynamic system, you must be able to influence the response. The actuator does this.
- Before choosing a specific actuator, consider which variables can be influenced.
4. **Make a linear model of the process, actuator, and sensor.**
   - Take the best choice for process, actuator, and sensor.
   - Identify the equilibrium point of interest.
   - Construct a small-signal dynamic model valid over the range of frequencies included in the performance specifications.
   - Validate this model with experimental measurements where possible.
   - Express the model in many forms: state-variable, pole-zero, and frequency response forms.
   - Simplify and reduce the order of the model, if necessary.
   - Quantify model uncertainty.
5. Make a simple trial design based on concepts of lead-lag compensation or PID control.

- To form an initial estimate of the complexity of the design problem, sketch a frequency-response (Bode) plot and a root locus plot with respect to plant gain.

- If the plant-actuator-sensor model is stable and minimum phase, the Bode plot will probably be the most useful; otherwise, the root locus shows very important information with respect to behavior in the right-half plane.

- Try to meet specifications with a simple controller of the lead-lag, PID variety.

- Do not overlook feedforward of the disturbances.

- Consider the effect of sensor noise.
6. **Consider modifying the plant itself for improved closed-loop control.**

- Based on the simple control design, evaluate the source of the undesirable characteristics of system performance.
- Reevaluate the specifications, the physical configuration of the process, and the actuator and sensor selections in light of the preliminary design. Return to step 1 if improvement seems necessary or feasible.
- It may be much easier to meet specifications by altering the process than to meet them by control strategies alone!
- Consider all parts of the design, not only the control logic, to meet the specifications in the most cost-effective way.
7. **Make a trial pole-placement design based on optimal control or other criteria.**

- If the trial-and-error compensators do not give entirely satisfactory performance, consider a design based on optimal control.
- Select the location for your control poles that balance system performance and control effort.
- Select the location for the estimator poles that represent a compromise between sensor and process noise.
- Plot the corresponding open-loop frequency response and the root locus to evaluate the stability margins of this design and its robustness to parameter changes.
- Compare this optimal design with the transform-method design and select the better of the two.
8. **Build a computer model and simulate the performance of the design.**

- After reaching the best compromise among process modification, actuator and sensor selection, and controller design choice, run a simulation of the system.
- Include important nonlinearities, parasitic effects, and parameter variations you expect to find during operation.
- Design iterations should continue until the simulation confirms acceptable stability and robustness.
- As the design progresses, more complete and detailed models ("truth models") will be used.
- If the performance is not satisfactory, return to step 1 and repeat. Consider modifying the plant itself for improved closed-loop control.
9. **Build a prototype and test it.**
   - At this point you verify the quality of the model, discover unexpected effects, and consider ways to improve the design.
   - Implement the controller using an embedded software/hardware.
   - Tune the controller, if necessary.
   - After these tests, you may want to reconsider the sensor, actuator, and process and return to step 1.
• This outline is an approximation of good practice.
• One very important consideration (Step 6) was for changing the plant itself to make the control problem easier and provide maximum closed-loop performance.
  – In many cases, proper plant modifications can provide additional damping or increase the stiffness, change in mode shapes, reduction of system response to disturbances, reduction of Coulomb friction, change in thermal capacity or conductivity, and so on.
  – Designing the system and “throwing it over the wall” to the control group is inefficient and flawed!
  – System design and control design must be done simultaneously!
Control System Design: Series vs. Parallel Compensation

Series Compensation

Parallel or Feedback Compensation
• The following factors must be considered when making a choice between series and parallel compensation:
  – Design procedures for a series compensator are more direct than those for a parallel compensator. The application of parallel compensators is sometimes more laborious, but it may be easier to implement.
  – Because of the physical form of the control system (e.g., electrical, hydraulic, mechanical) a series or parallel compensator may not exist or be practical.
  – The type of signal input to the compensator must be considered.
– The economics in the use of either technique for a given control system involves items such as size, weight, and cost of components and amplifiers. In the forward path the signal goes from a low- to a high-energy level (an amplifier for gain and/or isolation is generally necessary), whereas the reverse is true in the feedback loop (an amplifier is generally not necessary). Also, size and weight may be different for the series and parallel compensators.

– The environmental conditions in which the feedback control system is to be utilized affect the accuracy and stability of the controlled quantity.

– The problem of noise within a control system may determine the choice of compensator.
– The time of response desired for a control system is a determining factor. Often a faster time of response can be achieved by the use of parallel compensation.

– Some systems require “tight-loop” stabilization to isolate dynamics of one portion of a control system from other portions of the complete system. This can be accomplished by introducing an inner feedback loop around the portion to be isolated.

– When the plant has a pair of dominant complex poles that yield the dominant poles of $C(s)/R(s)$, the simple first-order series compensators (lead, lag, lead-lag) provide minimal improvement to the system’s time-response characteristics. For this situation, parallel compensation is more effective and is therefore more desirable.
Also, the available components and the designer’s experience and preferences influence the choice between a series and parallel compensator for achieving the desired tracking performance.
Digital Control of Dynamic Systems

- Digital Computer
  - Input: Digital Set Point
  - Input: Sampled & Quantized Measurement
  - Output: Sampled & Quantized Control Signal

- A/D Converter
- D/A Converter
- Sampling System
- Anti-Aliasing Filter
- Sensor
- Plant
- Final Control Element

Modeling, Analysis, & Control of Dynamic Systems

K. Craig 180
Advantages of Digital Control

- The current trend toward using dedicated, microprocessor-based, and often decentralized (distributed) digital control systems in industrial applications can be rationalized in terms of the major advantages of digital control:
  - Digital control is less susceptible to noise or parameter variation in instrumentation because data can be represented, generated, transmitted, and processed as binary words, with bits possessing two identifiable states.
Very high accuracy and speed are possible through digital processing. Hardware implementation is usually faster than software implementation.

Digital control can handle repetitive tasks extremely well, through programming.

Complex control laws and signal conditioning methods that might be impractical to implement using analog devices can be programmed.

High reliability can be achieved by minimizing analog hardware components and through decentralization using dedicated microprocessors for various control tasks.
– Large amounts of data can be stored using compact, high-density data storage methods.
– Data can be stored or maintained for very long periods of time without drift and without being affected by adverse environmental conditions.
– Fast data transmission is possible over long distances without introducing dynamic delays, as in analog systems.
– Digital control has easy and fast data retrieval capabilities.
– Digital processing uses low operational voltages (e.g., 0 - 12 V DC).
– Digital control has low overall cost.
Digital Signals are:
- discrete in time
- quantized in amplitude

You must understand the effects of:
- sample period
- quantization size

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<th>Discrete in Time</th>
<th>Continuous in Time</th>
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<tr>
<td>Discrete in Amplitude</td>
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<td>D-C</td>
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<tr>
<td>Continuous in Amplitude</td>
<td>C-D</td>
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• In a real sense, the problems of analysis and design of digital control systems are concerned with taking into account the effects of the sampling period, $T$, and the quantization size, $q$.

• If both $T$ and $q$ are extremely small (i.e., sampling frequency 50 or more times the system bandwidth with a 16-bit word size), digital signals are nearly continuous, and continuous methods of analysis and design can be used.

• It is most important to understand the effects of all sample rates, fast and slow, and the effects of quantization for large and small word sizes.

• It is worthy to note that the single most important impact of implementing a control system digitally is the delay associated with the D/A converter, i.e., $T/2$. 
Simulation of Continuous and Quantized Signal

![Graph showing simulation of continuous and quantized signal. The x-axis represents time in seconds, ranging from 0 to 1. The y-axis represents amplitude, with values ranging from 0 to 1.4. The graph displays a continuous line and a quantized line, illustrating the difference in representation.]
Continuous Output and D/A Output
Aliasing

- The analog feedback signal coming from the sensor contains useful information related to controllable disturbances (relatively low frequency), but also may often include higher frequency "noise" due to uncontrollable disturbances (too fast for control system correction), measurement noise, and stray electrical pickup. Such noise signals cause difficulties in analog systems and low-pass filtering is often needed to allow good control performance.
In digital systems, a phenomenon called *aliasing* introduces some new aspects to the area of noise problems. If a signal containing high frequencies is sampled too infrequently, the output signal of the sampler contains low-frequency ("aliased") components not present in the signal before sampling. If we base our control actions on these false low-frequency components, they will, of course, result in poor control. The theoretical absolute minimum sampling rate to prevent aliasing is 2 samples per cycle; however, in practice, rates of about 10 are more commonly used. A high-frequency signal, inadequately sampled, can produce a reconstructed function of a much lower frequency, which can not be distinguished from that produced by adequate sampling of a low-frequency function.
Simulation of Continuous and Sampled Signal
- Shown is an example of a computer-controlled motion control system.

\[ M = 0.001295 \text{ lbf-s}^2/\text{in} \]
\[ B = 0.259 \text{ lbf-s/in} \]
\[ K_{mf} = 0.5 \text{ lbf/A} \]
\[ K_{pa} = 2.0 \text{ A/V} \]

Sensor Gain = 1.0 V/in

Computational Delay = 0.008 s

pure/ideal mass and damper